Dispersive Wave Equations on Networks

ADITYA KARKERA

Capstone Final Report for BSc (Honours) in

Mathematical, Computational and Statistical Sciences

Supervised by: Dr. David Smith

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Abstract

Mathematical, Computational, and Statistical Sciences

BSc (Hons)

Dispersive Wave Equations on Networks

by Aditya KARKERA

Interface problems for dispersive wave equations define a class of problems involving higher spatial order initial boundary value problems. Such problems are either impossible or infeasibly difficult to solve using classical methods such as Fourier transform pairs. In this paper, we study the use of a new method for solving such problems, the unified transform method, which involves the synthesis rather than separation of variables. We study a full implementation of the UTM on the half-line heat problem and partial implementations on simple interface problems for the linear schrödinger and linearised Korteweg-De vries equations.

Keywords: *Unified transform method, evolution partial differential equations, interface initial boundary value problems*

Contents

List of Figures

Chapter 0

The Unified Transform Method

The object of this paper is to study interface problems for partial differential equations, particularly the Linear Schrödinger and Linearised Korteweg-de Vries equations, and derive solution representations for them. The Unified Transform Method will be integral to this study and derivation since classical methods are unable to do the same. This chapter outlines the problem we are interested in, the method we will use to study this problem, and the organisation of this paper for this study.

0.1 Introduction

Interface problems for dispersive partial differential equations (PDEs) are a class of initial boundary value problems (IBVPs) where the solution of a dispersive equation in one domain prescribes boundary conditions for the equations in adjacent domains (Pinsky, 2011). Such problems model various physical phenomenon across quantum mechanics, heat flow, and physics: some dispersive equations this paper in particular will touch on. These problems, however, rarely allow for explicit closed-form solutions with uniform convergence at the boundaries. Additionally, such problems cannot generally be solved using classical analytic Fourier methods, especially for dispersive equations of higher spatial order.

0.1.1 Dispersive Wave Equations

A time-dependent, scalar, linear partial differential equation with constant coefficients on an unbounded space domain admits the following plane wave solutions: $u(x,t) = e^{i(kx+\omega t)}, k \in \mathbb{R}$, where k is the wavenumber and ω is the frequency. Since not all values of k can be taken in the plane wave solutions for each value of ω, the linear PDE imposes a dispersion relation $\omega = \omega(k)$ (Trefethen, 1994). Wave equations with such dispersive relations, which therefore disperse different wavelengths at different phase velocities over time in some spatial domain, are dispersive wave equations. In this paper, we will study two dispersive wave equations, namely:

Linear Schrödinger Equation (LS)

The linear Schrödinger equation is a linear partial differential equation that models the wave function of a quantum-mechanical system. The full problem we are interested in is outlined in Section 3.1 but the equation of interest is

$$[\partial_t + i\partial_{xx}]q(x,t) = 0.$$

Linearised Korteweg-de Vries Equation (LKdV)

The linearised Kortweg-de Vries equation is the linear form of the non-linear Kortweg-de Vries equation, which models shallow water waves and is an actively studied PDE. The full problem we are interested in is outlined in Section 4.1 but the equation of interest is:

$$[\partial_t - \partial_{xxx}]q(x,t) = 0 \implies [\partial_t + i(-i\partial_{xxx})]q(x,t) = 0.$$

Note that both of the equations of interest involve the imaginary unit i.

0.1.2 Interface Problems

Interface problems involve boundary conditions for equations in adjacent domains prescribed by equation solutions in a given domain. For example, a network of interconnected rods on which heat flow is modelled (Sheils and Smith, 2015). Determining the well-posedness of such problems is a non-trivial issue and boundary conditions must be derived to be imposed at the interface. This is beyond the scope of this paper and imposed boundary conditions will be stated as part of the problem definition. We will study interface problems for PDEs involving the two dispersive wave equations described above.

0.2 The Inadequacy of Classical Methods

Our study, which involves interface problems for dispersive wave equations, does not lend itself to easy analysis using classical methods. This is only true for the LKdV equation, which is of a higher spatial order than manageable for most classical transforms (> 2). Primarily, classical methods involve the separation of variables and application of classical *transform pairs* such as the Fourier transforms, which include the sine, cosine, and other transforms (Fokas and Smith, 2016). These pairs are often deduced by inspecting the PDE but can be systematically derived for some IBVPs using Green's functions but, crucially, not for all of them. Consider the solution via sine transform pairs for the heat equation ($\partial_t - \partial_{xx} = 0$) on the half-line:

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \sin(\lambda x) e^{-\lambda^2 t}\left[\int_0^\infty \sin(\lambda s) u_0(s) ds + \lambda \int_0^t e^{\lambda^2 \tau} g_0(\tau) d\tau\right] d\lambda, \tag{1}$$

where $0 < x < \infty$, $t > 0$, u_0 is the initial datum ($u(x,0) = u_0$), and g_0 is a boundary datum. Note that the problem has a separable solution form ($u(x,t) = X(x)T(t)$) in functions of x and t as follows:

$$\frac{X''}{X} = \frac{T'}{T}.$$

Rewriting these as ODEs after equating the constant ratios above with $-\lambda^2$ gives us $X''(x) + \lambda^2 X(x) = 0$ and $T'(t) + \lambda^2 T(t) = 0$. The particular solutions for these ODEs are $e^{i\lambda x}$ and $^{-\lambda^2 t}$. From the separable solution form $(u(x,t) = X(x)T(t))$, we can infer that a particular solution for the heat equation is given by $U(\lambda)e^{i\lambda x - \lambda^2 t}$ which is the same as stating $u(x,t) = \int U(\lambda)e^{i\lambda x - \lambda^2 t} d\lambda$. The Ehrenpreis principle allows that the solution for this formulation of the heat equation can indeed be expressed in this way, the intuition being that we integrate the solutions across the

λ domain to approach a general solution.

Notice that the solution representation in (1) is not of this convenient Ehrenpreis form. Since $\sin(\lambda x) = 0$ when $x = 0$, it is also not apparent if $u(0, t) = g_0(t)$. The integral at $x = 0$ also cannot be freely swapped for $\lim_{x\to 0}$ of the integral because $\sin 0 = 0$. Thus, (1) is not uniformly convergent at $x = 0$ unless the boundary condition is homogeneous. The lack of uniform convergence means that numerical evaluation of (1) is infeasible. In fact, most classical methods are defined for the homogeneous cases of inhomogeneous problems and suffer the same drawback. This especially complicates the study of the higher order, dispersive equations whose solutions we must numerically evaluate to model. This is assuming we have found the correct transform to use, which in itself is a computationally expensive and laborious task (Smith, 2019).

But now consider the Fourier transform pair solution to the initial value problem (not IBVP) of the heat equation:

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t} \hat{u_0}(\lambda) d\lambda, \tag{2}$$

where $-\infty < x < \infty$, $t > 0$, and $\hat{u_0}$ is the Fourier transform of the initial datum u_0. We observe that this is of the Ehrenpreis form described, with a specified contour and function $U(\lambda)$. If we could replicate this solution form in a way that were uniformly convergent, of the Ehrenpreis form, and standard enough to avoid having to guess the correct transform, we would much more easily be able to study interface problems for dispersive wave equations. This is why we use the Unified Transform Method.

0.3 The Unified Transform Method

The Unified Transform Method (UTM), a relatively new transform (Fokas, 1997), extends and augments the classical transform approaches by working towards the *synthesis*, rather than *separation*, of variables. Through the use of tools from complex analysis, linear algebra, real analysis, and classical methods such as Fourier transforms, the UTM has been proven to yield explicit solution representations that are uniformly convergent, hence producing new solution formulae for problems already solved by classical methods and even producing formulae for problems with no classical approach (Sheils, 2015). This method enables us to study interface problems for dispersive wave equations where classical methods would fail to. Its implementation can be broken down into three stages (Smith, 2019), as detailed below and followed for the rest of this paper. More generalised frameworks also exist (Deconinck, Trogdon, and Vasan, 2014; Fokas and Pelloni, 2015; Fokas, 2008).

0.3.1 Stage I

Assume that the given problem has a solution. Then work to derive:

Global Relation

A relation of the eventual solution to the boundaries of the space-time domain, derived from an application of Green's theorem to a so-called local relation (Deconinck, Trogdon, and Vasan, 2014).

Ehrenpreis Form

A representation of the eventual solution in terms of complex contour integrals of initial datum and boundary value transforms

0.3.2 Stage II

Hold assumption that solution exists. Construct a Data to Unknown map to cast unknown terms in terms of the problem data. Then use the global relation derived in Stage I and boundary values defined to substitute in the Ehrenpreis form. This gives us a unique integral representation of the solution defined only with problem data.

0.3.3 Stage III

Having found a unique solution representation, we work to prove existence. This is straightforward in principle and involves treating the solution representation from Stage II as a function where we show that q satisfies our representation. This proves existence.

0.4 Organisation of Paper

This paper is organised into four chapters:

1. **Chapter 1** introduces the various mathematical tools, ranging from complex analysis to linear algebra, that will be employed in the implementation of the three-stage UTM described above.

2. **Chapter 2** demonstrates an outline implementation of the UTM for the half-line Dirichlet heat problem in the non-interface or two-point case. This chapter sketches all three stages to demonstrate a successful UTM solution to a canonical PDE.

3. **Chapter 3** extends the UTM to the simple interface LS problem involving an interface with three domains. The first two stages of the UTM are implemented, with the third stage outside of the scope of this paper.

4. **Chapter 4** extends the UTM to the simple interface LKdV problem involving an interface with three domains. The first two stages of the UTM are implemented, with the third stage outside of the scope of this paper.

5. **Chapter 5** briefly concludes and describes next steps for this study.

Chapter 1

Toolkit

A successful application of the UTM requires the use of several tools to work with. This chapter outlines these tools for later reference.

1.1 Fourier Transforms

Definition 1.1.1 (Fourier Transform) *For some complex-valued function $f(x)$, $x \in \mathbb{R}$, the Fourier transform $\hat{f}(\lambda)$ is given by*

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx,$$

for $\lambda \in \mathbb{C}$

Definition 1.1.2 (Inverse Fourier Transform) *For some complex-valued Fourier Transform $\hat{f}(\lambda)$, $\lambda \in \mathbb{C}$, the inverse Fourier transform of $\hat{f}(\lambda)$ is, for $x \in \mathbb{R}$,*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda x} dx.$$

1.2 Real Analysis

Definition 1.2.1 (Uniform Convergence) *A sequence of functions $f_n : X \to Y$ converges uniformly if $\forall \epsilon > 0$, $\exists N_\epsilon \in \mathbb{N}$ such that $\forall n \geq N_\epsilon$ and $\forall x \in X$ one has $|f_n(x) - f(x)| < \epsilon$.*

Theorem 1.2.1 (Uniform Convergence in Derivatives) *Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of differentiable functions whose derivatives f_n' are continuous. If f_n converges uniformly to f and f_n' converges uniformly to g, then the limit f is differentiable and its derivative is $f' = g$. That is to say,*

$$\lim_{n \to \infty} \frac{d f_n(x)}{dx} = \frac{d \lim_{n \to \infty} f_n(x)}{dx}.$$

Theorem 1.2.2 (Ehrenpreis's Fundamental Principle) *Every solution of a system of homogeneous partial differential equations with constant coefficients can be represented as the integral with respect to an appropriate Radon measure over the complex characteristic variety of the system (Farkas et al., 2013).*

Remark 1 *The Ehrenpreis Fundamental Principle is a result from analysis with a more complicated statement as well as an accompanying, more rigorously-proven theorem (Ehrenpreis-Palamodov Theorem). For our purposes, the statement above conveys the important fact that solutions of well-posed homogeneous linear PDEs with constant coefficients can be expressed in the form of a contour integral that involves particular solutions across some complex variable domain (since the Principle applies to bounded, smooth, convex, domains).*

1.3 Complex Analysis

Theorem 1.3.1 (Cauchy's Integral Theorem) *(Asmar and Grafakos, 2018) Let $U \subseteq \mathbb{C}$ be a simply connected open set, and let $f : U \to \mathbb{C}$ be a holomorphic (or complex analytic) function. Let $\gamma : [a, b] \to U$ be a smooth closed curve. Then,*

$$\oint_{\gamma} f(z)dz = 0.$$

Theorem 1.3.2 (Jordan's Lemma) *If $V \subseteq \mathbb{C}$ is an open set, $\forall R > 0$, $V \cap B(0, R)$ is the union of finitely many simply connected regions, let $C_R^{\pm} := C(0, R) \cap (V \cap \mathbb{C}^{\pm})$ and if $(f : \text{clos}(V) \to \mathbb{C})$ is continuous with*

$$\lim_{R\to\infty} (\max\{f(\lambda) : \lambda \in C_R^{\pm}\}) = 0,$$

then, $\forall a > 0$,

$$\lim_{R\to\infty} \int_{C_R^{\pm}} e^{\pm ia\lambda} f(\lambda)d\lambda = 0.$$

Corollary 1.3.2.1 *If $f : U \to \mathbb{C}$ is analytic, U is open and simply connected set containing clos(V), and $f(\lambda) \to 0$ uniformly in arg(λ) as $\lambda \to \infty$ within clos(V), then, $\forall a > 0$,*

$$\lim_{R\to\infty} \int_{\partial V \cap B(0,R) \cap \mathbb{C}^{\pm}} e^{\pm ia\lambda} f(\lambda)d\lambda = 0.$$

1.4 Linear Algebra

Theorem 1.4.1 (Cramer's Rule) *Suppose that $Ax = b$ is a system of linear equations where A is an $n \times n$ matrix and*

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Let A_j denote the matrix that is obtained by taking the j^{th} column of A and replacing it with the column matrix b. If $det(A) \neq 0$, then the system has a unique solution given by

$$x_1 = \frac{det(A_1)}{det(A)}, \quad x_2 = \frac{det(A_2)}{det(A)}, \quad \ldots, \quad x_n = \frac{det(A_n)}{det(A)}.$$

Chapter 2

The Heat Equation

2.1 A Summary of the UTM in Practice

In this chapter, we briefly go though a full implementation of the UTM on the heat equation. The aim is to help the reader familiarise with the main components of the UTM in this two-point case before a more thorough implementation (up to Stage II) in the interface problems for LS and LKdV. This is an outline with most working excluded.

2.2 Defining the Problem

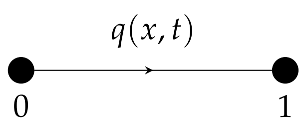

FIGURE 2.1: Spatial Domain for (Half-Line) Heat Problem

We are interested in applying the UTM to analyse the Half-Line Dirichlet Heat Problem, which is defined by the following Partial Differential Equation (PDE), an Initial Condition (IC), and two Boundary Conditions

(BC(1) and BC(2)).

$$[\partial_t - \partial_{xx}]q(x,t) = 0 \tag{2.PDE}$$

$$q(x,0) = q_0(x) \tag{2.IC}$$

$$q(0,t) = g_0 \tag{2.BC (1)}$$

$$q(1,t) = g_1 \tag{2.BC (2)}$$

2.3 Stage I

2.3.1 Global Relation

We now proceed to explore the first part of Stage I of the UTM, where we seek to use the (2.PDE) and the (2.IC) to extract a Global Relation from the Problem as stated. Assuming $\exists q : [0,1] \times [0,T]$ satisfying the PDE and IC, we apply the Fourier transform to both sides of the PDE and then intergate by parts. This gives us an ODE which we can integrate in t to solve for $\hat{q}(\lambda; t)$. This is done below.

$$0 = e^{\lambda^2 t}\hat{q}(\lambda;t) - \hat{q}_0(\lambda;0) + \int_0^t e^{\lambda^2 s}\left(\partial_x q(0,s) + i\lambda q(0,s)\right) ds \\ - e^{-i\lambda}\int_0^t e^{\lambda^2 s}\left(\partial_x q(1,s) + i\lambda q(1,s)\right) ds \tag{2.1}$$

For convenience, we denote the above using the following notation (which will be analysed later in the method as well and is of importance).

$$f_j(\lambda; X; t) := \int_0^t e^{\lambda^2 s}\partial_x^j q(X,s) ds \tag{2.2}$$

so that (2.1) is written as

$$\hat{q_0}(\lambda) - e^{\lambda^2 t}\hat{q}(\lambda;t) = i\lambda f_0(\lambda;0;t) + f_1(\lambda;0;t) - e^{-i\lambda}(f_1(\lambda;1;t) + i\lambda f_0(\lambda;1;t)) \tag{2.GR}$$

Equation (2.GR) above is the global relation.

2.3.2 Setting Up Contours

Having derived a global relation from the PDE and IC, we now work to set up contours in the complex plane as a first step to derive the Ehrenpreis form equation, which will be crucial in deriving our final solution representation. First, however, we solve for $q(x,t)$ in order to set up the integrals for contour deformation in the complex plane. We achieve this with an inverse Fourier transform to the global relation.

$$2\pi q(x,t) = \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t}\hat{q}(\lambda)d\lambda$$

$$- \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t}(i\lambda f_0(\lambda;0;t) + f_1(\lambda;0;t))d\lambda \tag{2.3}$$

$$+ \int_{-\infty}^{\infty} e^{i\lambda(x-1) - \lambda^2 t}(f_1(\lambda;1;t) + i\lambda f_0(\lambda;1;t))d\lambda \tag{2.4}$$

Now, our aim is to deform the latter two contours of integration from the above ((2.3) and (2.4)) away from $\mathbb{R}$. In order to accomplish this, we first define the closed sectors within the complex plane that we seek to deform these contours within.

Definition 2.3.1 *(Closed Sectors of Interest)*

$$C^{\pm} := \{\lambda \in \mathbb{C} : \pm\Im(\lambda) > 0\}$$
$$D := \{\lambda \in \mathbb{C} : \Re(\lambda^2) < 0\}, \qquad D^{\pm} := D \cap \mathbb{C}^{\pm}$$
$$E := \{\lambda \in \mathbb{C} : \Re(\lambda^2) > 0\}, \qquad E^{\pm} := E \cap \mathbb{C}^{\pm}$$

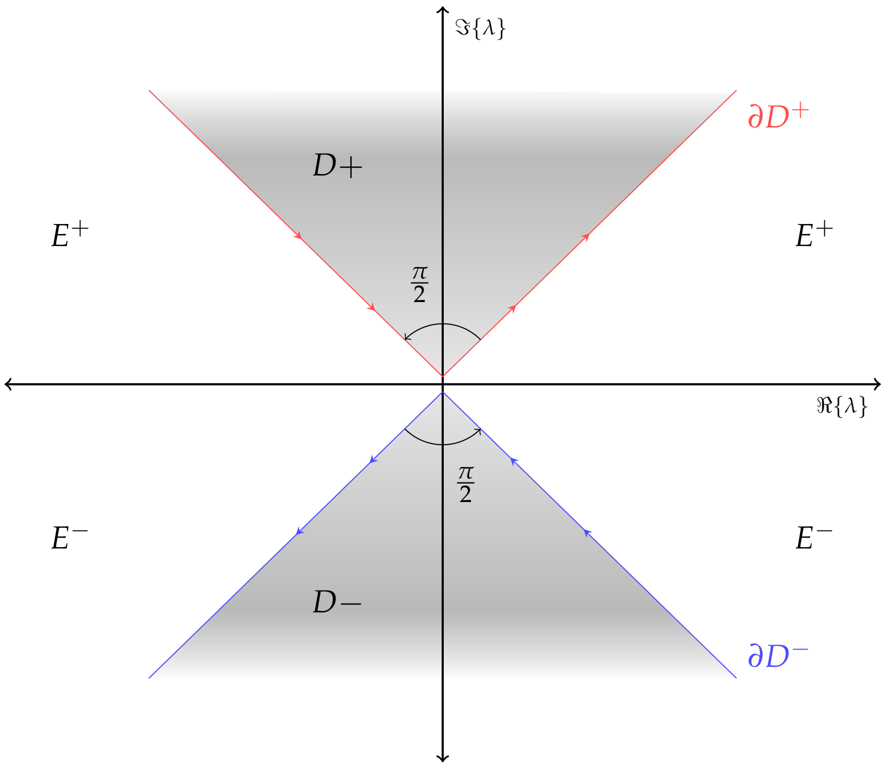

FIGURE 2.2: Closed Sectors for Contour Deformation (Heat Equation)

Analytically, by studying the equation introduced in (2.2), we find that $e^{-\lambda^2 t}(i\lambda f_j(\lambda; X, t) + \lambda f_j(\lambda; X, t)) = \mathcal{O}(|\lambda^{-2}|)$ uniformly in $\arg(\lambda)$ as $\lambda \to \infty$ within $\operatorname{clos}(E)$. This decay satisfies the conditions for an application of Jordan's Lemma to the integrands we are interested in. By Jordan's Lemma

$$\int_{\partial E^+} e^{i\lambda x - \lambda^2 t} \left(i\lambda f_0(\lambda; 0, t) + f_1(\lambda; 0, t)\right) d\lambda = 0$$

Substituting this implication back into our inverse Fourier transform representation of $q(x,t)$ by altering (2.3) and (2.4) (we subtract $\int_{D^{\pm}}$ from these integrals) gives us the following.

$$\begin{aligned} 2\pi q(x,t) = &\int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2 t}\hat{q}(\lambda)d\lambda \\ &- \int_{\partial D^+} e^{i\lambda x-\lambda^2 t}(i\lambda f_0(\lambda;0;t)+f_1(\lambda;0;t))d\lambda \\ &- \int_{\partial D^-} e^{i\lambda(x-1)-\lambda^2 t}(f_1(\lambda;1;t)+i\lambda f_0(\lambda;1;t))d\lambda, \end{aligned} \qquad (2.EF_t)$$

valid for $(x,t) \in (0,1)\times[0,T]$. We have thus arrived at the Ehrenpreis Form in t for the Heat Equation and have concluded Stage I of the UTM. It is possible, and preferable, to express ($2.EF_t$) in terms of some τ, $\forall\tau \in [t,T]$. We do this by employing a similar argument to our application of Jordan's Lemma previously. This is outlined in Appendix A.

2.4 Stage II

2.4.1 Summary

In Stage II, we incorporate the boundary conditions ((2.BC (1)) and (2.BC (2))) into the global relation we derived. We then take advantage of the fact that $f_j(\lambda;X,\tau)$ depends on λ entirely through λ^2 in $e^{\lambda^2\tau}$ to set up a linear system of two equations in two unknowns. We take $e^{\lambda^2\tau}\hat{q}(\lambda;\tau)$ to be "known" until we vanish terms containing it later. After using Cramer's rule, we vanish $e^{\lambda^2\tau}\hat{q}(\lambda;\tau)$ terms by analytically showing decay in those terms and applying Jordan's lemma. This gives us the solution representation for the heat equation using the UTM:

2.4.2 Solution Representation

$$
\begin{aligned}
2\pi q(x,t) = &\int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2 t}\hat{q}(\lambda)d\lambda \\
&- \int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\left((i\lambda h_0(\lambda;\tau) + \frac{\zeta^+(\lambda;M(\cdot;\tau)+\hat{q_0})}{\Delta(\lambda)}\right) d\lambda \\
&- \int_{\partial D^-} e^{i\lambda(x-1)-\lambda^2 t}\left((i\lambda h_1(\lambda;\tau) + \frac{\zeta^-(\lambda;M(\cdot;\tau)+\hat{q_0})}{\Delta(\lambda)}\right) d\lambda
\end{aligned}
\tag{2.SRT_τ}
$$

Where h_j, M, $\hat{q_0}$, $\zeta^{\pm}$ are explicitly defined in the problem data. $\zeta^{\pm}$ refer to the Cramer rule representations of the unknowns, and $\Delta(\lambda)$ is the determinant of the system.

2.5 Stage III

Having derived a solution representation, we define $q(x,t)$ using (2.SRT_τ) and verify if it indeed solves the problem we defined in Stage I by checking if it satisfies the PDE, IC, and BCs.

2.5.1 PDE

Any $(x,t) \in (0,1) \times (0,T)$ has a closed neighbourhood Ω within $(0,1) \times (0,T)$ such that $e^{i\lambda x'-\lambda^2 t'} \to 0$ exponentially uniformly on $(x',t') \in \Omega$ as $\lambda \to \infty$ along $\mathbb{R}$ or $\partial D^{\pm}$. Therefore, all partial derivatives of q exist and are given by differentiating the integrand. Taking the derivative terms as defined in (2.SRT_τ) above and integrating therefore does satisfy the PDE.

2.5.2 Initial Condition

We aim to isolate the initial conditions from our (2.SRT_τ). Before we accomplish this, note that $\forall T \in (SR_T)$, T can be replaced by τ, $\forall \tau \in [t, T]$. So, q is equivalently defined by both (SR_T) and ((2.SRT_τ)). When we set $\tau = t$ and $t = 0$, $h_j(\lambda; 0) = 0 \implies M(\lambda; 0) = 0$. Thus vanish these terms and we are left with a solution representation in terms of $\hat{q}_0$ which we reduce to an inverse Fourier transform using Jordan's lemma.

$$\implies 2\pi q(x, 0) = q_0(x), \forall x \in (0, 1)$$

We have thus shown that (2.SRT_τ) does satisfy (2.IC).

2.5.3 Boundary Conditions

Demonstrating a full implementation of Stage III is beyond the scope of this paper, but for the heat equation this involves the setting up of a series representation of the solution. We then check if the solution satisfies homogeneous and inhomogeneous boundary conditions using this series representation and construct an inverse Fourier transform of a fourier transform for the latter (Smith, 2019).

We have demonstrated that the solution representation satisfies the PDE and IC. In addition, it has been shown in Appendix Appendix A that it satisfies the BCs as well. In conclusion, the solution representation solves the problem defined and we have successfully employed the UTM to solve the heat equation.

Chapter 3

The Linear Schrödinger Equation

3.1 Defining the Problem

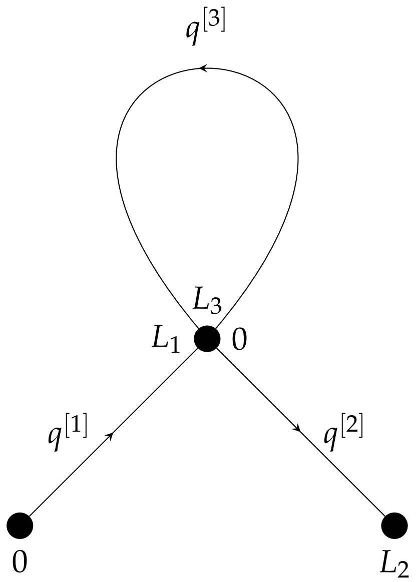

FIGURE 3.1: Simple Interface Domain with Single Interface

We are interested in applying the UTM to analyse the Time-Dependent, Zero Potential Linear Schrödinger Equation, which is defined by the following Partial Differential Equation (PDE), an Initial Condition (IC), and two Boundary Conditions (BC(1) and BC(2)). We will study this equation across a simple interface problem involving one interface and three domains. This involves conditions for continuity at the interface (Continuity) and conservation of flux at the interface (Conservation).

$$[\partial_t - i(-\partial_{xx})]q^{[j]}(x,t) = 0 \tag{3.PDE}$$

$$q^{[j]}(x,0) = q_0^{[j]}(x) \tag{3.IC}$$

$$q^{[1]}(0,t) = 0 \tag{3.BC (1)}$$

$$q^{[2]}(L_2,t) = 0 \tag{3.BC (2)}$$

$$q^{[1]}(L_1,t) = q^{[2]}(0,t) = q^{[3]}(0,t) = q^{[3]}(L_3,t) \tag{3.Continuity}$$

$$\partial_x q^{[1]}(L_1,t) + \partial_x q^{[3]}(L_3,t) = \partial_x q^{[2]}(0,t) + \partial_x q^{[3]}(0,t) \tag{3.Conservation}$$

Figure 3.1 is the physical domain for which the problem is defined.

3.1.1 Preliminary Work

First, we investigate the Fourier transform $\hat{\cdot}$ with $i\frac{d^2}{dx^2}$ on $\mathbb{C}^\infty[0,L_j]$.

$$\widehat{\left(i\frac{d}{dx}\right)^2 \phi(\lambda)} = i\int_0^{L_j} e^{-i\lambda x}\phi''(x)dx$$

Integrating the above by parts in x

$$\begin{aligned}
&= i\left([e^{-i\lambda x}(\phi'(x) + i\lambda\phi(x))]_{x=0}^{x=L_j} - \lambda^2 \int_0^{L_j} e^{-i\lambda x}\phi(x)dx \right) \\
&= i\left((e^{-i\lambda L_j}(\phi'(L_j) + i\lambda\phi(L_j))) - (\phi'(0) + i\lambda\phi(0)) - \lambda^2\hat{\phi}(\lambda) \right) \\
&= e^{-i\lambda L_J}\left(i\phi'(L_j) - \lambda\phi(L_j)\right) - \left(i\phi'(0) - \lambda\phi(0)\right) - i\lambda^2\hat{\phi}(\lambda)
\end{aligned}$$

3.2 Stage I

3.2.1 Global Relation

We now proceed to explore the first part of Stage I of the UTM, where we seek to use the (3.PDE) and the (3.IC) to extract a Global Relation from the problem as stated. Assuming $\exists q^{[j]} : [0, L_j] \times [0, T]$ satisfying the PDE and IC, we apply the Fourier transform to both sides of the PDE:

$$\begin{aligned} 0 &= \widehat{[\partial_t + i\partial_{xx}]} q^{[j]}(\lambda; t) \\ &= \widehat{\partial_t} q^{[j]}(\lambda; t) + \widehat{i\partial_{xx}} q^{[j]}(\lambda; t) \\ &= \frac{d}{d_t}\hat{q}^{[j]}(\lambda; t) + e^{-i\lambda L_j}(i\phi'(L_j) - \lambda\phi(L_j)) - (i\phi'(0) - \lambda\phi(0)) - i\lambda^2\hat{\phi}(\lambda) \\ &= \left[\frac{d}{d_t} - i\lambda^2\right]\hat{q}^{[j]}(\lambda; t) + e^{-i\lambda L_j}(i\partial_x q^{[j]}(L_j, t) - \lambda q^{[j]}(L_j, t)) \\ &\quad - (i\partial_x q^{[j]}(0, t) - \lambda q^{[j]}(0, t)) \end{aligned} \tag{3.1}$$

We multiply both sides by the Integrating Factor $e^{-i\lambda^2 t}$

$$\begin{aligned} &= \frac{d}{d_t}\left[e^{-i\lambda^2 t}\hat{q}^{[j]}(\lambda; t)\right] + e^{-i\lambda L_j - i\lambda^2 t}(i\partial_x q^{[j]}(L_j, t) - \lambda q^{[j]}(L_j, t)) \\ &\quad - e^{-i\lambda^2 t}(i\partial_x q^{[j]}(0, t) - \lambda q^{[j]}(0, t)) \end{aligned} \tag{3.2}$$

Where equation (3.2) above is an ODE. We integrate the above in t and use the IC to solve the ODE for $\hat{q}^{[j]}(\lambda; t)$.

$$\begin{aligned} 0 = e^{-i\lambda^2 t}\hat{q}^{[j]}(\lambda; t) - \hat{q}_0^{[j]}(\lambda; 0) + e^{-i\lambda L_j}\int_0^t e^{-i\lambda^2 s}(i\partial_x q^{[j]}(L_j, s) \\ - \lambda q^{[j]}(L_j, s) ds - \int_0^t e^{-i\lambda^2 s}(i\partial_x q^{[j]}(0, s) - \lambda q^{[j]}(0, s)) ds \end{aligned} \tag{3.3}$$

For convenience, we denote the above using the following notation (which will be analysed later in the method as well and is of importance),

$$f_k^{[j]}(\lambda; X; t) := \int_0^t e^{-i\lambda^2 s} \partial_x^k q^{[j]}(X, s) ds \tag{3.4}$$

so that (3.3) is written as

$$\begin{aligned} \hat{q_0}^{[j]}(\lambda) - e^{-i\lambda^2 t} \hat{q}^{[j]}(\lambda; t) = e^{-i\lambda L_j} [i f_1^{[j]}(\lambda; L_j; t) - \lambda f_0^{[j]}(\lambda; L_j; t)] \\ - (i f_1^{[j]}(\lambda; 0; t) - \lambda f_0^{[j]}(\lambda; 0; t)) \end{aligned} \tag{3.GR}$$

valid $\forall \lambda \in \mathbb{C}, \forall t \in [0, T]$. Equation (3.GR) above is the global relation. Note the intervals we are operating in: $[0, L_1], [0, L_2], [0, L_3]$. We therefore derive three global relations, one for each j.

3.2.2 Setting Up Contours

Having derived a global relation from the PDE and IC, we now work to set up contours in the complex plane as a first step to derive the Ehrenpreis form equation, which will be crucial in deriving our final solution representation. First, however, we solve for $q^{[j]}(x, t)$ in order to set up the equation for contour deformation in the complex plane. We achieve this with an inverse Fourier transform applied to the global relation. Note:

$$\begin{aligned} \hat{q_0}^{[j]}(\lambda) - e^{-i\lambda^2 t} \hat{q}^{[j]}(\lambda; t) = (\dots) \\ \implies \hat{q}^{[j]}(\lambda; t) = e^{i\lambda^2 t} \left[\hat{q^{[j]}}(\lambda) - (\dots) \right] \end{aligned}$$

We apply the inverse Fourier transform to both sides of this rearrangement and so find

$$
\begin{aligned}
2\pi q^{[j]}(x,t) = &\int_{-\infty}^{\infty} e^{i\lambda x + i\lambda^2 t}\hat{q_0}^{[j]}(\lambda)d\lambda \\
&-\int_{-\infty}^{\infty} e^{i\lambda(x-L_j)+i\lambda^2 t}(if_1^{[j]}(\lambda;L_j;t) - \lambda f_0^{[j]}(\lambda;L_j;t))d\lambda \qquad (3.5)\\
&+\int_{-\infty}^{\infty} e^{i\lambda x+i\lambda^2 t}(if_1^{[j]}(\lambda;0;t) - \lambda f_0^{[j]}(\lambda;0;t))d\lambda \qquad (3.6)
\end{aligned}
$$

Now, our aim is to deform the latter two contours of integration from the above ((3.5) and (3.6)) away from $\mathbb{R}$. In order to accomplish this, we first define the closed sectors within the complex plane that we seek to deform these contours into.

Definition 3.2.1 *(Closed Sectors of Interest)*

$$
\begin{aligned}
&C^{\pm} := \{\lambda \in \mathbb{C} : \pm\Im(\lambda) > 0\} \\
&D := \{\lambda \in \mathbb{C} : \Re(-i\lambda^2) < 0\}, \quad D^{\pm} := D \cap \mathbb{C}^{\pm} \\
&E := \{\lambda \in \mathbb{C} : \Re(-i\lambda^2) > 0\}, \quad E^{\pm} := E \cap \mathbb{C}^{\pm}
\end{aligned}
$$

We now explore the limiting properties of the notation we introduced in (3.4) as observed in the latter two integrals of (3.5) and (3.6). We do this by integrating by parts in s.

$$
\begin{aligned}
e^{i\lambda^2 t} f_k^{[j]}(\lambda;X,t) &= \int_0^t e^{i\lambda^2(t-s)}\partial_x^k q^{[j]}(X,s)ds \\
&= \underbrace{i\lambda^{-2}[e^{i\lambda^2(t-s)}\partial_x^k q^{[j]}(X,s)]_{s=0}^{s=t}}_{\mathcal{O}(|\lambda^{-2}|)} - \underbrace{i\lambda^{-2}\int_0^t e^{i\lambda^2(t-s)}\partial_t\partial_x^k q^{[j]}(X,s)ds}_{\mathcal{O}(|\lambda^{-2}|)} \\
&= \mathcal{O}(|\lambda^{-2}|), \text{uniformly in } \arg(\lambda) \text{ as } \lambda \to \infty \text{ within } \operatorname{clos}(E)
\end{aligned}
$$

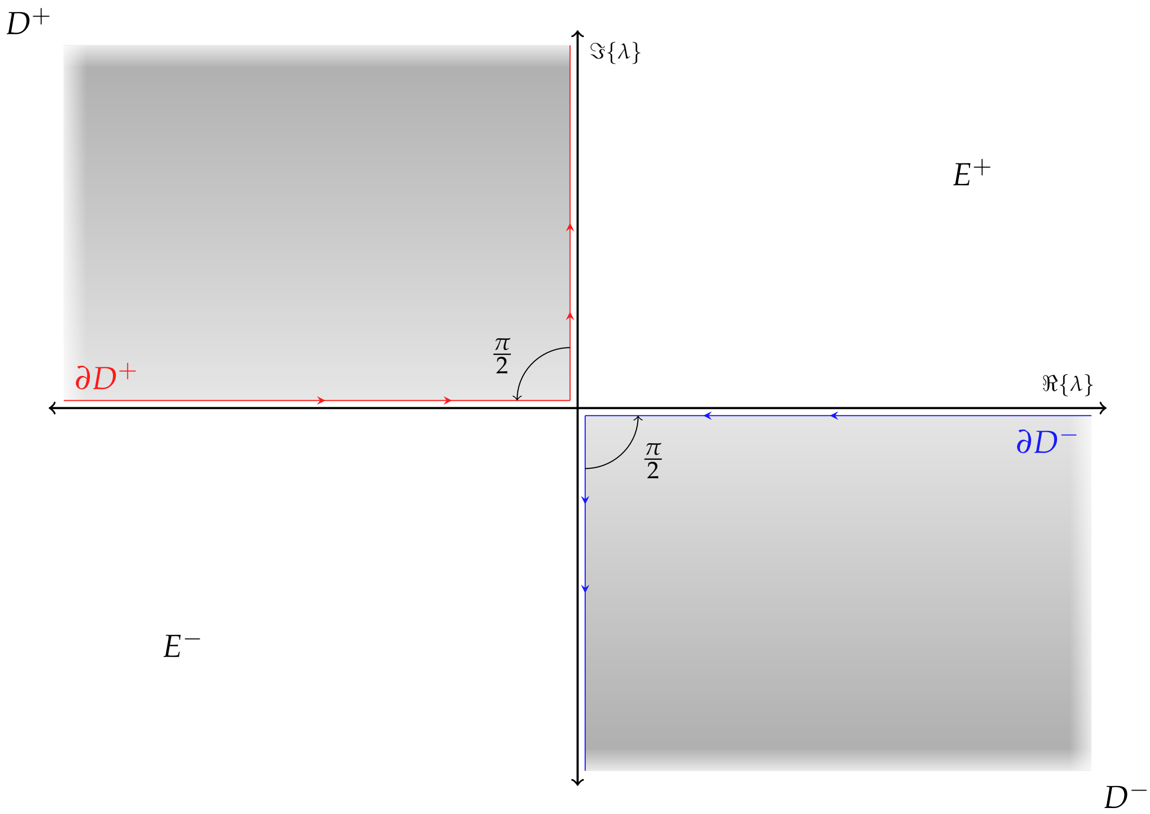

FIGURE 3.2: Closed Sectors for Contour Deformation (LS Equation)

The above can be applied to find that $e^{i\lambda^2 t}(if_j(\lambda; X, t) - \lambda f_j(\lambda; X, t)) = \mathcal{O}(|\lambda^{-1}|)$ uniformly in $\arg(\lambda)$ as $\lambda \to \infty$ within $\operatorname{clos}(E)$. Having shown that the integrands decay, we are prepared to apply Jordan's Lemma to (3.5) and (3.6). By Jordan's Lemma, for $x \in (0, L_j)$,

$$\int_{\partial E^+} e^{i\lambda(x-L_j)+i\lambda^2 t}(if_1^{[j]}(\lambda; L_j; t) - \lambda f_0^{[j]}(\lambda; L_j; t))d\lambda = 0,$$
$$\int_{\partial E^+} e^{i\lambda x+i\lambda^2 t}(if_1^{[j]}(\lambda; 0; t) - \lambda f_0^{[j]}(\lambda; 0; t))d\lambda = 0.$$

This gives us, for the integral in (3.6)

$$\int_{-\infty}^{\infty} \dots d\lambda = \int_{-\infty}^{\infty} - \int_{\partial E^+} \dots d\lambda = \int_{\partial D^+} \dots d\lambda, \tag{3.7}$$

and, for the integral in (3.5)

$$\begin{aligned} \int_{-\infty}^{\infty} \dots d\lambda &= -\int_{\infty}^{-\infty} \dots d\lambda \\ &= -\int_{\infty}^{-\infty} - \int_{\partial E^-} \dots d\lambda = -\int_{\partial D^-} \dots d\lambda. \end{aligned} \tag{3.8}$$

Substituting this implication back into our inverse Fourier transform representation of $q^{[j]}(x,t)$ by altering (3.5) and (3.6) yields

$$\begin{aligned} 2\pi q^{[j]}(x,t) = &\int_{-\infty}^{\infty} e^{i\lambda x + i\lambda^2 t} \hat{q_0}^{[j]}(\lambda) d\lambda \\ &+ \int_{\partial D^-} e^{i\lambda(x-L_j)+i\lambda^2 t}(if_1^{[j]}(\lambda;L_j;t) - \lambda f_0^{[j]}(\lambda;L_j;t))d\lambda \\ &+ \int_{\partial D^+} e^{i\lambda x + i\lambda^2 t}(if_1^{[j]}(\lambda;0;t) - \lambda f_0^{[j]}(\lambda;0;t))d\lambda, \end{aligned} \tag{3.EF_t}$$

valid for $(x,t) \in (0, L_j) \times [0, T]$. We have thus arrived at the Ehrenpreis form in t for the Linear Schrödinger equation on an interface. But we can make eventual computation easier by expressing EF_t in terms of some τ, $\forall \tau \in [t, T]$. We do this by employing a similar argument to our application of Jordan's Lemma previously.

$$ie^{i\lambda^2 t} \int_t^{\tau} e^{\lambda^2 s} q^{[j]}(X,s)ds + \lambda e^{i\lambda^2 s} \int_t^{\tau} e^{i\lambda^2 s} \partial_x q^{[j]}(X,s)ds) = \mathcal{O}(|\lambda|^{-1}),$$

uniformly in $\arg(\lambda)$ as $\lambda \to \infty$ within $\operatorname{clos}(D)$. We can use this information to recast EF_t in terms of τ as below

$$\begin{aligned} 2\pi q^{[j]}(x,t) = &\int_{-\infty}^{\infty} e^{i\lambda x + i\lambda^2 t} \hat{q_0}^{[j]}(\lambda) d\lambda \\ &+ \int_{\partial D^-} e^{i\lambda(x-L_j)+i\lambda^2 t}(if_1^{[j]}(\lambda; L_j; \tau) - \lambda f_0^{[j]}(\lambda; L_j; \tau))d\lambda \\ &+ \int_{\partial D^+} e^{i\lambda x + i\lambda^2 t}(if_1^{[j]}(\lambda; 0; \tau) - \lambda f_0^{[j]}(\lambda; 0; \tau))d\lambda, \quad (3.EF_\tau) \end{aligned}$$

valid for $(x,t) \in (0, L_j) \times [0, \tau], \tau \in [0, T]$. We have so arrived at the Ehrenpreis form in τ (EF_τ) for the Linear Schrödinger equation on an interface and have concluded Stage I of the UTM.

3.3 Stage II

3.3.1 Progress Thus Far

We utilised the PDE and Initial Condition to derive first a global relation in terms of transforms of the PDE and IC. We then worked to find a representation (the Ehrenpreis Form) of the solution, q, in terms of contour integrals deformed away from $\mathbb{R}$. Until now, we have only required the PDE and IC. In Stage II, we include BCs.

3.3.2 Incorporating Boundary Conditions

Assume that $q^{[j]}$ satisfies not only the PDE and IC but also BC(1) and BC(2). Observe thus that

$$\underbrace{f_0^{[1]}(\lambda;0,\tau)}_{\text{Known Data}} = \int_0^\tau e^{-i\lambda^2 s} q^{[1]}(0,s)ds = \int_0^\tau e^{-i\lambda^2 s} 0 ds = 0, \tag{3.9}$$

$$\underbrace{f_0^{[2]}(\lambda;L_2,\tau)}_{\text{Known Data}} = \int_0^\tau e^{-i\lambda^2 s} q^{[2]}(L_2,s)ds = \int_0^\tau e^{-i\lambda^2 s} 0 ds = 0. \tag{3.10}$$

Applying BC(1) and BC(2) to the global relation gives us the following altered global relations

$$\hat{q_0}^{[1]}(\lambda) - e^{-i\lambda^2 t}\hat{q^{[1]}}(\lambda;t) = e^{-i\lambda}\left(if_1^{[1]}(\lambda;L_1;t) - \lambda f_0^{[1]}(\lambda;L_1;t)\right) - \left(if_1^{[1]}(\lambda;0;t) - 0\right) \tag{3.11}$$

$$\hat{q_0}^{[2]}(\lambda) - e^{-i\lambda^2 t}\hat{q^{[2]}}(\lambda;t) = e^{-i\lambda}\left(if_1^{[2]}(\lambda;L_2;t) - 0\right) - \left(if_1^{[2]}(\lambda;0;t) - \lambda f_0^{[2]}(\lambda;0;t)\right) \tag{3.12}$$

$$\hat{q_0}^{[3]}(\lambda) - e^{-i\lambda^2 t}\hat{q^{[3]}}(\lambda;t) = e^{-i\lambda}\left(if_1^{[3]}(\lambda;L_3;t) - \lambda f_0^{[3]}(\lambda;L_3;t)\right) - \left(if_1^{[3]}(\lambda;0;t) - \lambda f_0^{[3]}(\lambda;0;t)\right) \tag{3.13}$$

where $f_1^{[1]}(\lambda;L_1;t) = f_1^{[2]}(\lambda;0,\tau) + f_1^{[3]}(\lambda;0,\tau) - f_1^{[3]}(\lambda;L_3,\tau)$, which we get from conservation of flux and continuity conditions mean that $f_0^{[1]}(\lambda;L_1;t) = \lambda f_0^{[2]}(\lambda;0;t) = \lambda f_0^{[3]}(\lambda;L_3;t) = \lambda f_0^{[3]}(\lambda;0;t)$. We use these conditions to construct a linear system of unknowns. This requires isolating the unknown spectral functions in terms of known data.

3.3.3 Linear System

Isolating unknown spectral functions

The above application of boundary conditions to the global relations leads to the following simplified system of equations where unknown terms are expressed in terms of "known" terms (for now, we shall take $e^{\lambda^2\tau}\hat{q}(\lambda;\tau)$ to be "known" until we deal with it momentarily):

$$\begin{aligned}
&ie^{-i\lambda}\left[f_1^{[2]}(\lambda;0,\tau)+f_1^{[3]}(\lambda;0,\tau)-f_1(\lambda;L_3,\tau)\right]-\lambda e^{-i\lambda}f_0(\lambda;L_1,\tau)\\
&\qquad\qquad\qquad\qquad -if_1^{[1]}(\lambda;0,\tau)=\hat{q_0}^{[1]}(\lambda)-e^{-i\lambda^2 t}\hat{q}^{[1]}(\lambda;t)\\
&ie^{-i\lambda}f_1(\lambda;L_2,\tau)-if_1^{[2]}(\lambda;0,\tau)+\lambda f_0(\lambda;L_1,\tau)=\hat{q_0}^{[2]}(\lambda)-e^{-i\lambda^2 t}\hat{q}^{[2]}(\lambda;t)\\
&ie^{-i\lambda}f_1(\lambda;L_3,\tau)-\lambda e^{-i\lambda}f_0(\lambda;L_1,\tau)-if_1^{[3]}(\lambda;0,\tau)+\lambda f_0(\lambda;L_1,\tau)\\
&\qquad\qquad\qquad\qquad =\hat{q_0}^{[3]}(\lambda)-e^{-i\lambda^2 t}\hat{q}^{[3]}(\lambda;t)
\end{aligned}$$

Note that in (3.3.3), $f_j(\lambda;X,\tau)$ depends on λ entirely through λ^2 in $e^{\lambda^2\tau}$. If we apply the identity mapping $\lambda\mapsto\lambda$ and the mapping $\lambda\mapsto-\lambda$ to the global relation,

$$(\mathrm{GR})\Big|_{\lambda\mapsto\lambda}\quad(\mathrm{GR})\Big|_{\lambda\mapsto-\lambda}$$

we get six linearly independent equations involving six unknowns:

$$\begin{pmatrix} ie^{-i\lambda L_1} & ie^{-i\lambda L_1} & -ie^{-i\lambda L_1} & -\lambda-i\lambda & -i & 0 \\ ie^{i\lambda L_1} & ie^{i\lambda L_1} & -ie^{i\lambda L_1} & \lambda-i\lambda & -i & 0 \\ -i & 0 & 0 & \lambda & 0 & ie^{-i\lambda L_2} \\ -i & 0 & 0 & -\lambda & 0 & ie^{i\lambda L_2} \\ 0 & -i & ie^{-i\lambda L_3} & -\lambda e^{-i\lambda L_3}+\lambda & 0 & 0 \\ 0 & -i & ie^{i\lambda L_3} & \lambda e^{i\lambda L_3}\lambda & 0 & 0 \end{pmatrix} \begin{pmatrix} f_1^{[1]}(\lambda;0,\tau) \\ f_1^{[3]}(\lambda;0,\tau) \\ f_1^{[3]}(\lambda;L_3,\tau) \\ f_0^{[1]}(\lambda;L_1,\tau) \\ f_1^{[1]}(\lambda;0,\tau) \\ f_1^{[2]}(\lambda;L_2,\tau) \end{pmatrix}$$
$$= \begin{pmatrix} \hat{q_0}^{[1]}(\lambda) & \hat{q_0}^{[1]}(-\lambda) & \hat{q_0}^{[2]}(\lambda) & \hat{q_0}^{[2]}(-\lambda) & \hat{q_0}^{[3]}(\lambda) & \hat{q_0}^{[3]}(-\lambda) \end{pmatrix}^T$$
$$- e^{-i\lambda^2\tau} \begin{pmatrix} \hat{q}^{[1]}(\lambda;\tau) & \hat{q}^{[1]}(-\lambda;\tau) & \hat{q}^{[2]}(\lambda;\tau) & \hat{q}^{[2]}(-\lambda;\tau) & \hat{q}^{[3]}(\lambda;\tau) & \hat{q}^{[3]}(-\lambda;\tau) \end{pmatrix}^T$$

This paper does not go beyond this phase of the UTM's implementation for this problem, but the next steps are similar to the non-interface problem. The above linear system will have to be solved using Cramer's rule, whereupon we will be able to replace unknowns in the Ehrenpreis form with expressions of unknown terms using known terms. Then, using Jordan's lemma, we vanish all terms involving the last set of unknowns ($\hat{q}^{[j]}(\lambda;\tau)$). This yields a solution representation which must then be verified in Stage III. The non-interface steps for the above are described in Appendix B.

Chapter 4

The LKdV Equation

4.1 Defining the Problem

We are interested in applying the UTM to analyse the Linearised Korteweg-De Vries Equation, which is defined by the following Partial Differential Equation (PDE), an Initial Condition (IC), and three Boundary Conditions (BC(1), BC(2), and BC(3)). We will study this equation across a simple interface problem involving one interface and three domains. This involves conditions for continuity at the interface (Interface Continuity) and continuity of derivatives at the interface (Derivative Continuity).

$$[\partial_t + (-i\partial_{xxx})]q^{[j]}(x,t) = 0 \tag{PDE}$$

$$q^{[j]}(x,0) = q_0^{[j]}(x) \tag{IC}$$

$$q^{[1]}(0,t) = 0 \tag{BC (1)}$$

$$q^{[2]}(L_2,t) = 0 \tag{BC (2)}$$

$$q_x^{[2]}(L_2,t) = 0 \tag{BC (3)}$$

$$q^{[1]}(L_1,t) = q^{[2]}(0,t) = q^{[3]}(0,t) = q^{[3]}(L_3,t)$$

(Interface Continuity)

$$\partial_x q^{[1]}(L_1,t) = \partial_x q^{[2]}(0,t) = \partial_x q^{[3]}(0,t) = \partial_x q^{[3]}(L_3,t)$$

(Derivative Continuity)

Figure 3.1 is the physical domain for which the problem is defined.

4.1.1 Preliminary Work

Consider that $[\partial_t - \partial_{xxx}]$ can be expressed as $[\partial_t + i(-i\partial_{xxx})]$. First, we investigate the Fourier Transform $\hat{\cdot}$ with $\left(-i\frac{d^3}{dx^3}\right)$ on $\mathbb{C}^\infty[0, L_j]$. We are interested in this relationship to understand how a Fourier transform may be applied to the PDE as a whole across the interface problem being studied.

$$\widehat{(-i\tfrac{d}{dx})^3\phi}(\lambda) = i\int_0^{L_j} e^{-i\lambda x}\phi'''(x)dx$$

Integrating by parts in x gives us

$$\begin{aligned}\widehat{(-i\tfrac{d}{dx})^3\phi}(\lambda) = e^{-i\lambda L_j}&\left(-i\phi''(L_j) + \lambda(\phi'(L_j)) + i\lambda^2(\phi(L_j))\right)\\ &- \left(-i\phi''(0) + \lambda\phi'(0) + i\lambda^2\phi(0)\right) + \lambda^3\hat{\phi}(\lambda)\end{aligned}$$

4.2 Stage I

4.2.1 Global Relation

We now proceed to explore the first part of Stage I of the UTM, where we seek to use the (PDE) and the (IC) to extract a global relation from the problem as stated. Assuming $\exists q^{[j]} : [0, L_j] \times [0, T]$ satisfying the PDE and IC, we apply the Fourier transform and integrating factor $e^{i\lambda^3 t}$ to both sides of the PDE:

$$0 = \frac{d}{dt}(e^{i\lambda^3 t}\hat{q}^{[j]}(\lambda;t)) + e^{-i\lambda L_j + i\lambda^3 t}\left(\partial_{xx}q^{[j]}(L_j,t) + i\lambda\partial_x q^{[j]}(L_j,t) - \lambda^2\partial_x q^{[j]}(L_j,t)\right) - e^{i\lambda^3 t}(\partial_{xx}q^{[j]}(0,t) + i\lambda\partial_x q^{[j]}(0,t) - \lambda^2 q^{[j]}(0,t)) \quad (4.1)$$

Where equation (4.1) above is an ODE. We integrate the above in t and use the IC to solve the ODE for $\hat{q}^{[j]}(\lambda;t)$.

$$0 = e^{i\lambda^3 t}\hat{q}^{[j]}(\lambda;t) - \hat{q_0}^{[j]}(\lambda;0) + e^{-i\lambda L_j}\int_0^t e^{i\lambda^3 s}(\partial_{xx}q^{[j]}(L_j,s) + i\lambda\partial_x q^{[j]}(L_j,s) - \lambda^2\partial_x q^{[j]}(L_j,s))ds - \int_0^t e^{i\lambda^3 s}(\partial_{xx}q^{[j]}(0,s) + i\lambda\partial_x q^{[j]}(0,s) - \lambda^2 q^{[j]}(0,s))ds \quad (4.2)$$

For convenience, we denote the above using the following notation (which will be analysed later in the method as well and is of importance)

$$f_k^{[j]}(\lambda;X;t) := \int_0^t e^{i\lambda^3 s}\partial_x^k q^{[j]}(X,s)ds \quad (4.3)$$

so that (4.2) is written as

$$\hat{q_0}^{[j]}(\lambda) - e^{i\lambda^3 t}\hat{q}^{[j]}(\lambda;t) = e^{-i\lambda L_j}(f_2^{[j]}(\lambda;L_j,t) + i\lambda f_1^{[j]}(\lambda;L_j,t) - \lambda^2 f_0^{[j]}(\lambda;L_j,t)) - (f_2^{[j]}(\lambda;0,t) + i\lambda f_1^{[j]}(\lambda;0,t) - \lambda^2 f_0^{[j]}(\lambda;0,t)) \tag{4.4}$$

Equation (4.4) above is the global relation. Note the intervals we are operating in: $[0, L_1], [0, L_2], [0, L_3]$. We therefore derive three global relations, one for each j.

4.2.2 Setting Up Contours

Having derived a global relation from the PDE and IC, we now work to set up contours in the complex plane as a first step to derive the Ehrenpreis form equation, which will be crucial in deriving our final solution representation. First, however, we solve for $q^{[j]}(x,t)$ in order to set up the equation for contour deformation in the complex plane. We achieve this with an inverse Fourier transform applied to the global relation. Note:

$$\begin{aligned} \hat{q^{[j]}}(\lambda) - e^{i\lambda^3 t}\hat{q^{[j]}}(\lambda;t) &= (\dots) \\ \implies \hat{q^{[j]}}(\lambda;t) &= e^{-i\lambda^3 t}\left[\hat{q^{[j]}}(\lambda) - (\dots)\right] \end{aligned}$$

We apply an inverse Fourier transform to both sides of this rearrangement and this gives us

$$2\pi q^{[j]}(x,t) = \int_{-\infty}^{\infty} e^{i\lambda x - i\lambda^3 t}\hat{q_0}^{[j]}(\lambda)d\lambda$$
$$- \int_{-\infty}^{\infty} e^{i\lambda(x-L_j)-i\lambda^3 t}(f_2^{[j]}(\lambda;L_j;t) + i\lambda f_1^{[j]}(\lambda;L_j;t)) - \lambda^2 f_0^{[j]}(\lambda;L_j;t))d\lambda \tag{4.5}$$
$$- \int_{-\infty}^{\infty} e^{i\lambda x - i\lambda^3 t}(f_2^{[j]}(\lambda;0;t) + i\lambda f_1^{[j]}(\lambda;0;t) - \lambda^2 f_0^{[j]}(\lambda;0;t))d\lambda \tag{4.6}$$

Now, our aim is to deform the latter two contours of integration from the above ((4.5) and (4.6)) away from $\mathbb{R}$. In order to accomplish this, we first define the closures within the complex plane that we seek to deform these contours within.

Definition 4.2.1 *(Complex Plane Sectors of Interest)*

$$C^{\pm} := \{\lambda \in \mathbb{C} : \pm Im(\lambda) > 0\}$$
$$D := \{\lambda \in \mathbb{C} : Re(i\lambda^3) < 0\}, \qquad D^{\pm} := D \cap \mathbb{C}^{\pm}$$
$$E := \{\lambda \in \mathbb{C} : Re(i\lambda^3) > 0\}, \qquad E^{\pm} := E \cap \mathbb{C}^{\pm}$$

We now explore the limiting properties of the notation we introduced in (4.3) as observed in the latter two integrals of (4.5) and (4.6). We do this by integrating by parts in s.

$$\begin{aligned} e^{-i\lambda^3 t} f_k^{[j]}(\lambda;X,t) &= \int_0^t e^{-i\lambda^3(s-t)}\partial_x^k q^{[j]}(X,s)ds \\ &= \underbrace{i\lambda^{-3}[e^{i\lambda^3(s-t)}\partial_x^k q^{[j]}(X,s)]_{s=0}^{s=t}}_{\mathcal{O}(|\lambda|^{-3})} - \underbrace{i\lambda^{-3}\int_0^t e^{i\lambda^3(s-t)}\partial_t\partial_x^k q^{[j]}(X,s)ds}_{\mathcal{O}(|\lambda|^{-3})} \\ &= \mathcal{O}(|\lambda|^{-3}), \text{uniformly in } \arg(\lambda) \text{ as } \lambda \to \infty \text{ within } \operatorname{clos}(E) \end{aligned}$$

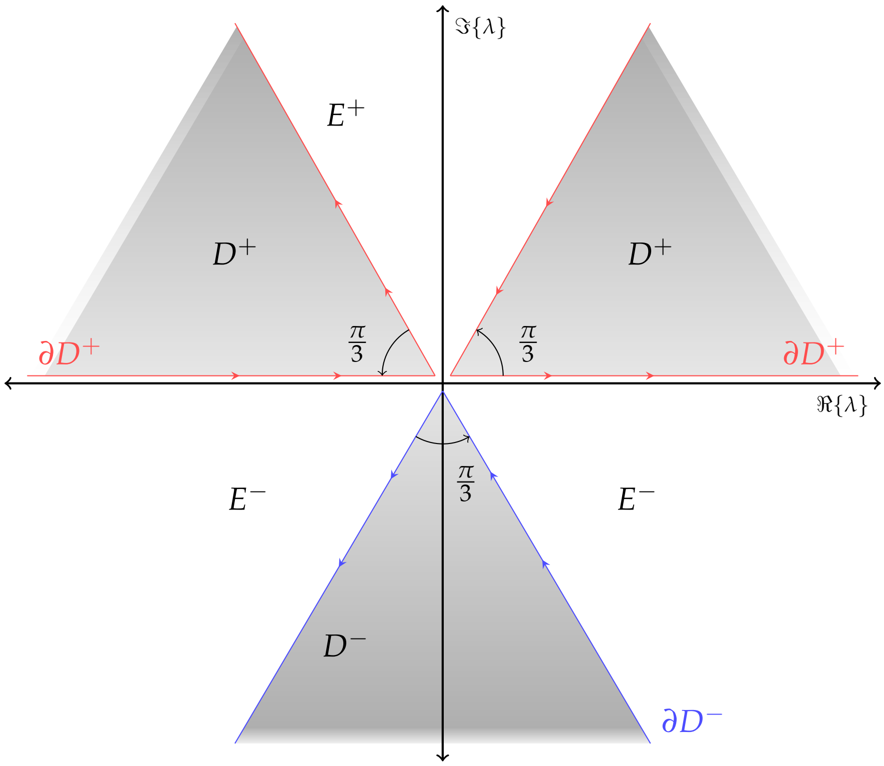

FIGURE 4.1: Closed Sectors for Contour Deformation (LKdV Equation)

The above can be applied to find that $e^{-i\lambda^3 t}(f_k^{[j]}(\lambda; X, t) + i\lambda f_k^{[j]}(\lambda; X, t) - \lambda^2 f_k^{[j]}(\lambda; X, t)) = \mathcal{O}(|\lambda|^{-1})$ uniformly in $\arg(\lambda)$ as $\lambda \to \infty$ within $\text{clos}(E)$ Having shown that the integrands decay, we are prepared to apply Jordan's lemma to (4.5) and (4.6). By Jordan's Lemma, for $x \in (0, L_j)$,

$$\int_{\partial E^+} e^{i\lambda(x-L_j)-i\lambda^3 t}(f_2^{[j]}(\lambda; L_j; t) + i\lambda f_1^{[j]}(\lambda; L_j; t) - \lambda^2 f_0^{[j]}(\lambda; L_j; t)d\lambda = 0,$$
$$\int_{\partial E^+} e^{i\lambda x-i\lambda^3 t}(f_2^{[j]}(\lambda; 0; t) + i\lambda f_1^{[j]}(\lambda; 0; t) - \lambda^2 f_0^{[j]}(\lambda; 0; t)d\lambda = 0.$$

This gives us, for the integral in (4.6)

$$\int_{-\infty}^{\infty} \ldots d\lambda = \int_{-\infty}^{\infty} - \int_{\partial E^+} \ldots d\lambda = \int_{\partial D^+} \ldots d\lambda, \tag{4.7}$$

and, for the integral in (3.5)

$$\int_{-\infty}^{\infty} \dots d\lambda = -\int_{\infty}^{-\infty} \dots d\lambda$$
$$= -\int_{\infty}^{-\infty} - \int_{\partial E^-} \dots d\lambda = -\int_{\partial D^-} \dots d\lambda. \tag{4.8}$$

Substituting this implication back into our inverse Fourier transform representation of $q^{[j]}(x,t)$ by altering (4.5) and (4.6) yields

$$\begin{aligned} 2\pi q^{[j]}(x,t) = &\int_{-\infty}^{\infty} e^{i\lambda x - i\lambda^3 t} \hat{q_0}^{[j]}(\lambda) d\lambda \\ &- \int_{\partial D^-} e^{i\lambda(x-L_j) - i\lambda^3 t} (f_2^{[j]}(\lambda; L_j; t) + i\lambda f_1^{[j]}(\lambda; L_j; t) - \lambda^2 f_0^{[j]}(\lambda; L_j; t)) d\lambda \\ &+ \int_{\partial D^+} e^{i\lambda x - i\lambda^3 t} (f_2^{[j]}(\lambda; 0; t) + i\lambda f_1^{[j]}(\lambda; 0; t) - \lambda^2 f_0^{[j]}(\lambda; 0; t)) d\lambda \end{aligned} \tag{4.EF_t}$$

valid for $(x,t) \in (0, L_j) \times [0, T]$.

We have thus arrived at the Ehrenpreis Form in t for the Linearised Korteweg-De Vries Equation. But we can make eventual computation easier by expressing EF_t in terms of some τ, $\forall \tau \in [t, T]$. We do this by employing a similar argument to our application of Jordan's Lemma previously.

$$e^{-i\lambda^3 t}\left(\int_t^\tau e^{i\lambda^3 s} \partial_{xx} q^{[j]}(X,s) ds + i\lambda \int_t^\tau e^{i\lambda^3 s} \partial_x q^{[j]}(X,s) ds - \lambda^2 \int_t^\tau e^{i\lambda^3 s} q^{[j]}(X,s) ds\right)$$
$$= \mathcal{O}(|\lambda|^{-1}) \tag{4.9}$$

uniformly in arg(λ) as $\lambda \to \infty$ within clos(D). We can use this information to recast EF_t in terms of τ as below

$$
\begin{aligned}
2\pi q^{[j]}(x,t) = &\int_{-\infty}^{\infty} e^{i\lambda x - i\lambda^3 t}\hat{q_0}^{[j]}(\lambda)d\lambda \\
&- \int_{\partial D^-} e^{i\lambda(x-L_j)-i\lambda^3 t}(f_2^{[j]}(\lambda;L_j;t) + \lambda f_1^{[j]}(\lambda;L_j;t) - i\lambda^2 f_0^{[j]}(\lambda;L_j;t))d\lambda \\
&+ \int_{\partial D^+} e^{i\lambda x - i\lambda^3 t}(f_2^{[j]}(\lambda;0;t) + i\lambda f_1^{[j]}(\lambda;0;t) - \lambda^2 f_0^{[j]}(\lambda;0;t))d\lambda
\end{aligned}
\tag{4.EF_τ}
$$

valid for $(x,t) \in (0,1) \times [0,\tau], \tau \in [0,T]$.

We have so arrived at the Ehrenpreis form in τ (EF_τ) for the linearised Korteweg-De Vries equation on an interface and have concluded Stage I of the UTM.

4.3 Stage II

4.3.1 Progress Thus Far

We utilised the PDE and Initial Condition to derive first a global relation in terms of transforms of the PDE and IC. We then worked to find a representation (the Ehrenpreis form) of this relation in terms of contour integrals deformed away from $\mathbb{R}$. Until now, we have only required the PDE and IC. In Stage II, we include BCs.

4.3.2 Incorporating Boundary Conditions

Assume that $q^{[j]}$ satisfies not only the PDE and IC but also BC(1), BC(2), and BC(3) Observe thus that

$$\underbrace{f_0^{[1]}(\lambda;0,\tau)}_{\text{Known Data}} = \int_0^\tau e^{i\lambda^3 s} q^{[1]}(0,s)ds = \int_0^\tau e^{i\lambda^3 s} 0 ds = 0, \tag{4.10}$$

$$\underbrace{f_0^{[2]}(\lambda;L_2,\tau)}_{\text{Known Data}} = \int_0^\tau e^{i\lambda^3 s} q^{[2]}(L_2,s)ds = \int_0^\tau e^{i\lambda^3 s} 0 ds = 0, \tag{4.11}$$

$$\underbrace{f_1^{[2]}(\lambda;L_2,\tau)}_{\text{Known Data}} = \int_0^\tau e^{i\lambda^3 s} \partial_x q^{[2]}(L_2,s)ds = \int_0^\tau e^{i\lambda^3 s} 0 ds = 0. \tag{4.12}$$

Applying BC(1), BC(2), and BC(3) to the global relation gives us the following altered global relations:

$$\begin{aligned}
\hat{q_0}^{[1]}(\lambda) - e^{i\lambda^3 t}\hat{q}^{[1]}(\lambda;\tau) &= e^{-i\lambda L_1}(f_2^{[1]}(\lambda;L_1,\tau) + i\lambda f_1^{[1]}(\lambda;L_1,\tau) - \lambda^2 f_0^{[1]}(\lambda;L_1,\tau)) \\
&\quad - (f_2^{[1]}(\lambda;0,\tau) + i\lambda f_1^{[1]}(\lambda;0,\tau) - 0 \\
\hat{q_0}^{[2]}(\lambda) - e^{i\lambda^3 t}\hat{q}^{[2]}(\lambda;\tau) &= e^{-i\lambda L_2}(f_2^{[2]}(\lambda;L_2,\tau) + 0 - 0) \\
&\quad - \left(f_2^{[2]}(\lambda;0,\tau) + i\lambda f_1^{[2]}(\lambda;0,\tau) - \lambda^2 f_0^{[2]}(\lambda;0,\tau)\right) \\
\hat{q_0}^{[3]}(\lambda) - e^{i\lambda^3 t}\hat{q}^{[3]}(\lambda;\tau) &= e^{-i\lambda L_3}\left(f_2^{[3]}(\lambda;L_3,\tau) + i\lambda f_1^{[3]}(\lambda;L_3,\tau) - \lambda^2 f_0^{[3]}(\lambda;L_3,\tau)\right) \\
&\quad - \left(f_2^{[3]}(\lambda;0,\tau) + i\lambda f_1^{[3]}(\lambda;0,\tau) - \lambda^2 f_0^{[3]}(\lambda;0,\tau)\right)
\end{aligned}$$

where it follows from interface continuity conditions that $f_0^{[1]}(\lambda;L_1,\tau) = f_0^{[2]}(\lambda;0,\tau) = f_0^{[3]}(\lambda;L_3,\tau) = f_0^{[3]}(\lambda;0,\tau)$ and it follows from derivatives continuity conditions that $f_1^{[1]}(\lambda;L_1,\tau) = f_1^{[2]}(\lambda;0,\tau) = f_1^{[3]}(\lambda;L_3,\tau) = f_1^{[3]}(\lambda;0,\tau)$. We are essentially able to take advantage of the interface nature of the problem to considerably simplify our work, and this will be a recurring feature of working in interfaces as seen below.

4.3.3 Linear System

Isolating unknown spectral functions

The above application of boundary conditions to the global relations leads to the following simplified system of equations where unknown terms are expressed in terms of "known" terms (for now, we shall take $e^{i\lambda^3\tau}\hat{q^{[j]}}(\lambda;\tau)$ to be "know" until we deal with it momentarily):

$$e^{-i\lambda L_1} f_2^{[1]}(\lambda; L_1, \tau) + i\lambda e^{-i\lambda L_1} f_1^{[1]}(\lambda; L_1, \tau) - \lambda^2 e^{-i\lambda L_1} f_0^{[1]}(\lambda; L_1, \tau)$$
$$- f_2^{[1]}(\lambda; 0, \tau) - i\lambda f_1^{[1]}(\lambda; 0, \tau) = \hat{q_0}^{[1]}(\lambda) - e^{i\lambda^3 t}\hat{q^{[1]}}(\lambda; \tau)$$

$$e^{-i\lambda L_2} f_2^{[2]}(\lambda; L_2, \tau) - f_2^{[2]}(\lambda; 0, \tau) - i\lambda f_1^{[1]}(\lambda; L_1, \tau) + \lambda^2 f_0^{[1]}(\lambda; L_1, \tau)$$
$$= \hat{q_0}^{[2]}(\lambda) - e^{i\lambda^3 t}\hat{q}^{[2]}(\lambda; \tau)$$

$$e^{-i\lambda L_3} f_2^{[3]}(\lambda; L_3, \tau) + i\lambda e^{-i\lambda L_3} f_1^{[1]}(\lambda; L_1, \tau) - \lambda^2 e^{-i\lambda L_3} f_0^{[1]}(\lambda; L_1, \tau)$$
$$- f_2^{[3]}(\lambda; 0, \tau) - i\lambda f_1^{[1]}(\lambda; L_1, \tau) + \lambda^2 f_0^{[1]}(\lambda; L_1, \tau) = \hat{q_0}^{[3]}(\lambda) - e^{i\lambda^3 t}\hat{q}^{[3]}(\lambda; \tau)$$

Note that in the system from (4.3.3), $f_j(\lambda; X, \tau)$ depends on λ entirely through λ^3 in $e^{i\lambda^3\tau}$. If we apply the mappings $\lambda \mapsto \lambda$, $\lambda \mapsto \alpha\lambda$, and $\lambda \mapsto \alpha^2\lambda$ $\left(\alpha \in \mathbb{C} = \sqrt[3]{1}\right)$ to the global relation (GR),

$$(\text{GR})\Big|_{\lambda\mapsto\lambda} \quad (\text{GR})\Big|_{\lambda\mapsto\alpha\lambda} \quad (\text{GR})\Big|_{\lambda\mapsto\alpha^2\lambda}$$

we get three linearly independent equations involving nine unknowns (after elementary row operations to simplify the matrix).

$$\begin{pmatrix} e^{-i\lambda L_1} & i\lambda e^{-i\lambda L_1} & -\lambda^2 e^{-i\lambda L_1} & -1 & -i\lambda & 0 & 0 & 0 & 0 \\ e^{-i\alpha\lambda L_1} & i\alpha\lambda e^{-i\alpha\lambda L_1} & -\alpha\lambda^2 e^{-i\alpha\lambda L_1} & -1 & -i\alpha\lambda & 0 & 0 & 0 & 0 \\ e^{-i\alpha^2\lambda L_1} & i\alpha^2\lambda e^{-i\alpha^2\lambda L_1} & -\alpha^2\lambda^2 e^{-i\alpha^2\lambda L_1} & -1 & -i\alpha^2\lambda & 0 & 0 & 0 & 0 \\ 0 & -i\lambda & \lambda^2 & 0 & 0 & e^{-i\lambda L_2} & -1 & 0 & 0 \\ 0 & -i\alpha\lambda & \alpha\lambda^2 & 0 & 0 & e^{-i\alpha\lambda L_2} & -1 & 0 & 0 \\ 0 & -i\alpha^2\lambda & \alpha^2\lambda^2 & 0 & 0 & e^{-i\alpha^2\lambda L_2} & -1 & 0 & 0 \\ 0 & i\lambda e^{-i\lambda L_3} & -\lambda^2 e^{-i\lambda L_3} & 0 & 0 & -e^{-i\lambda L_2} & 1 & e^{-i\lambda L_3} & -1 \\ 0 & i\alpha\lambda e^{-i\alpha\lambda L_3} & -\alpha\lambda^2 e^{-i\alpha\lambda L_3} & 0 & 0 & -e^{-i\alpha\lambda L_2} & 1 & e^{-i\alpha\lambda L_3} & -1 \\ 0 & i\alpha^2\lambda e^{-i\alpha^2\lambda L_3} & -\alpha^2\lambda^2 e^{-i\alpha^2\lambda L_3} & 0 & 0 & e^{-i\alpha^2\lambda L_2} & 1 & e^{-i\alpha^2\lambda L_3} & -1 \end{pmatrix}$$

$$\begin{pmatrix} f_2^{[1]}(\lambda; L_1, \tau) \\ f_1^{[1]}(\lambda; L_1, \tau) \\ f_0^{[1]}(\lambda; L_1, \tau) \\ f_2^{[1]}(\lambda; 0, \tau) \\ f_1^{[2]}(\lambda; 0, \tau) \\ f_2^{[2]}(\lambda; L_2, \tau) \\ f_2^{[2]}(\lambda; 0, \tau) \\ f_2^{[3]}(\lambda; L_3, \tau) \\ f_2^{[3]}(\lambda; 0, \tau) \end{pmatrix} = \begin{pmatrix} \hat{q_0}^{[1]}(\lambda) \\ \hat{q_0}^{[1]}(\alpha\lambda) \\ \hat{q_0}^{[1]}(\alpha^2\lambda) \\ \hat{q_0}^{[2]}(\lambda) \\ \hat{q_0}^{[2]}(\alpha\lambda) \\ \hat{q_0}^{[2]}(\alpha^2\lambda) \\ \hat{q_0}^{[3]}(\lambda) \\ \hat{q_0}^{[3]}(\alpha\lambda) \\ \hat{q_0}^{[3]}(\alpha^2\lambda) \end{pmatrix} - e^{i\lambda^3\tau} \begin{pmatrix} \hat{q}^{[1]}(\lambda; \tau) \\ \hat{q}^{[1]}(\alpha\lambda; \tau) \\ \hat{q}^{[1]}(\alpha^2\lambda; \tau) \\ \hat{q}^{[2]}(\lambda; \tau) \\ \hat{q}^{[2]}(\alpha\lambda; \tau) \\ \hat{q}^{[2]}(\alpha^2\lambda; \tau) \\ \hat{q}^{[3]}(\lambda; \tau) \\ \hat{q}^{[3]}(\alpha\lambda; \tau) \\ \hat{q}^{[3]}(\alpha^2\lambda; \tau) \end{pmatrix}$$

Similar to LS, this paper ends the implementation of the UTM for this problem here, with similar next steps regarding the linear system, solution representation, and Stage III. These are described in detail for the non-interface LKdV problem in Appendix C.

Chapter 5

Conclusion

5.1 What This Paper Did

In this project, we explored the UTM, studied its full implementation on the half-line heat problem and partial implementations for the LS and LKdV equations. For the latter two, we implemented the method to extract global relation and Ehrenpreis form formulae. We observed with the linear system that the problem becomes computationally expensive very quickly and requires a solution representation that can be numerically evaluated by a computer. But we noticed that the UTM's implementation is standard throughout and does in fact produce global relations and Ehrenpreis form equations for further analysis. While our partial implementation stopped with the linear systems, we laid out the data for a solution representation and isolated unknowns in terms of known data.

5.2 What This Paper Didn't Do

This project did not fully implement the UTM for the LS and LKdV equations. It did not provide solution representations, let alone verify these representations in Stage III. These are obvious next steps. Finally, without verified solution representations, it is impossible to numerically evaluate the solution(s). Another next step would thus be to use software to numerically evaluate the verified solution representations, check the LS solution against the solution from classical methods, and potentially visualise the solutions.

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Appendix A

Heat Equation Appendix

A.0.1 Stage I EF_τ

We can make eventual computation easier by expressing EF_t in terms of some τ, $\forall\tau \in [t, T]$. We do this by employing a similar argument to our application of Jordan's Lemma previously.

$$e^{-\lambda^2 t}(i\lambda \int_t^\tau e^{\lambda^2 s} q(X,s)ds + \int_t^\tau e^{\lambda^2 s}\partial_x q(X,s)ds) = \mathcal{O}(|\lambda|^{-1}),$$

uniformly in $arg(\lambda)$ as $\lambda \to \infty$ within $clos(D)$.

We can use this information to recast EF_t in terms of τ as below

$$\begin{aligned}2\pi q(x,t) = &\int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2 t}\hat{q}(\lambda)d\lambda \\ &- \int_{\partial D^+} e^{i\lambda x-\lambda^2 t}(i\lambda f_0(\lambda;0;\tau) + f_1(\lambda;0;\tau))d\lambda \\ &- \int_{\partial D^-} e^{i\lambda(x-1)-\lambda^2 t}(f_1(\lambda;1;\tau) + i\lambda f_0(\lambda;1;\tau))d\lambda, \qquad (2.EF_\tau)\end{aligned}$$

valid for $(x,t) \in (0,1) \times [0,T], \tau \in [0,T]$.

We have so arrived at the Ehrenpreis Form in τ (EF_τ).

A.0.2 Stage III Boundary Conditions

Similar to our work with the initial condition above, we aim to isolate (2.BC (1)) and (2.BC (2)) within our derived (2.SRT_τ). To do this, we need to reconstruct (SR_τ) as a series representation.

Series Representation

We first use Jordan's Lemma to deform the part of $\int_{\partial D^\pm}$ dependent on $\hat{q_0}$ to the real line perturbed along semicircular contours pf radius ϵ around the poles of $\Delta(\lambda)$, $(k\pi, k \in \mathbb{Z})$. We also deform $\int_{-\infty}^{\infty} e^{i\lambda x}\hat{q_0}(\lambda)d\lambda$ "up" around each zero of $\Delta(\lambda)$, to $\Gamma^\pm$ Observe that

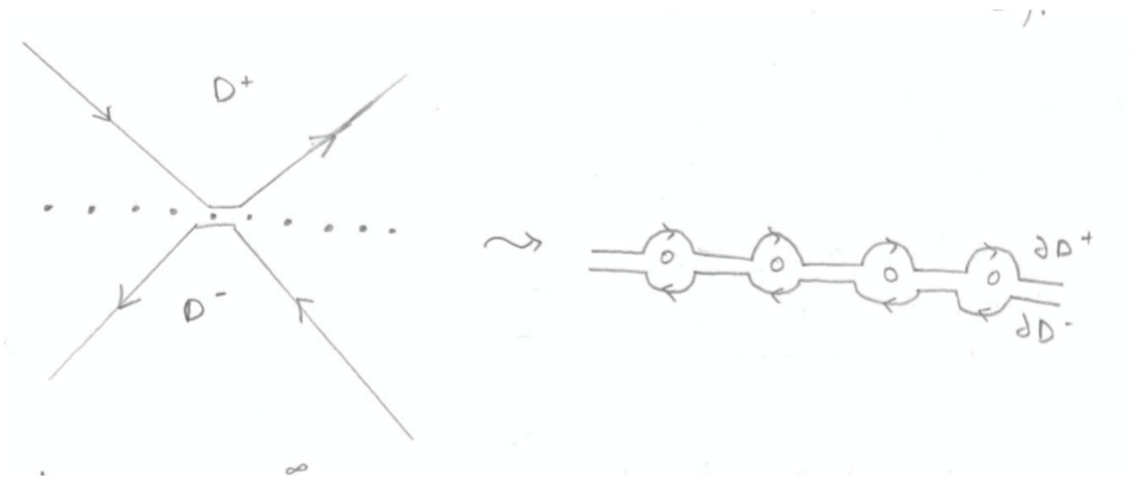

FIGURE A.1: Heat Equation Pole Perturbation

$$\hat{q_0}(\lambda) = \frac{\zeta^+(\lambda;\hat{q_0}) - e^{-i\lambda}\zeta^-(\lambda;\hat{q_0})}{\Delta(\lambda)} \tag{A.1}$$

We use (A.1) above in conjunction with the deformations described above to combine the integrals along $\Gamma^\pm$ into a series of integrals about small circular contours. This yields

$$\begin{aligned}2\pi q(x,t) = &\sum_{k\in\mathbb{Z}} \int_{C(k\pi,\epsilon)} e^{i\lambda(x-1)-\lambda^2 t}\left(\frac{-\hat{q_0}(\lambda)+\hat{q_0}(-\lambda)}{e^{-i\lambda}-e^{i\lambda}}\right) d\lambda \\ &- \int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\left((i\lambda h_0(\lambda;\tau)+\frac{\zeta^+(\lambda;M(\cdot;\tau))}{\Delta(\lambda)}\right) d\lambda \\ &- \int_{\partial D^-} e^{i\lambda(x-1)-\lambda^2 t}\left((i\lambda h_1(\lambda;\tau)+\frac{\zeta^-(\lambda;M(\cdot;\tau))}{\Delta(\lambda)}\right) d\lambda\end{aligned} \qquad (2.SRT_\tau - Series)$$

Homogeneous Boundary Conditions

Studying ($2.SRT_\tau - Series$), we notice two things of importance: first, regardless if $x = 0$ or $x = 1$, residues cancel out since $\int_{C(k\pi,)} \dots d\lambda = -\int_{C(-k\pi,)} \dots d\lambda$. This is akin to eliminating the first integral as a self-cancelling sum of circular integrals around poles that mirror each other (for every $k\pi$, there is an equivalent $-k\pi$). Taking this fact together with the second and third integrals evaluating to 0 if (2.BC (1)) and (2.BC (2)) are homogeneous, we can conclude that ($2.SRT_\tau$) satisfies the Boundary Conditions for the Heat Equation provided they are homogeneous.

Inhomogeneous Boundary Conditions

If (2.BC (1)) and (2.BC (2)) are inhomogeneous, the fact remains that regardless if $x = 0$ or $x = 1$, residues cancel out since $\int_{C(k\pi,)} \dots d\lambda = -\int_{C(-k\pi,)} \dots d\lambda$. Now, for each case $x = 0$ and $x = 1$, we aim to show that the remainder of ($2.SRT_\tau - Series$) yields the Boundary Conditions. We begin with the case where $x = 0$. We change variables in $\int_{\partial D^-} \dots d\lambda$

such that $\lambda \mapsto -\lambda$. We combine both integrals and simplify to see that

$$\begin{aligned}
2\pi q(0,t) &= -\int_{\partial D^+} e^{-\lambda^2 t}[i\lambda h_0(\lambda;\tau) + e^{i\lambda} i\lambda h_1(\lambda;\tau) + i\lambda\{h_0(\lambda;\tau) - e^{i\lambda}h_1(\lambda;\tau)\}]d\lambda \\
&= -\int_{\partial D^+} e^{-\lambda^2 t} 2i\lambda h_0(\lambda;\tau)d\lambda \\
&= -\int_{\partial D^+} e^{-\lambda^2 t} 2i\lambda \int_0^T e^{\lambda^2 s} g_0(s)dsd\lambda
\end{aligned}$$

Interestingly, the form of the above expression resembles an inverse Fourier transform followed by a Fourier transform, with the only discrepancy being the exponential terms. We try to see if we can substitute the relevant variables to acquire the proper form. Change variables once more: let $\lambda^2 = -i\rho \implies \lambda = i\sqrt{i\rho}$. This does produce a constructed Inverse Fourier Transform and notice also that $\frac{d\rho}{d\lambda} = 2i\lambda$ Thus,

$$-\int_{-\mathbb{R}} e^{i\rho t} \int_0^T e^{-i\rho s} g_0 ds d\rho = 2\pi g_0(t)$$

We now check the case where $x = 1$. We change variables in $\int_{\partial D^+} \ldots d\lambda$ such that $\lambda \mapsto -\lambda$. We combine both integrals and simplify to see that

$$\begin{aligned}
2\pi q(1,t) = -\int_{\partial D^-} & e^{-\lambda^2 t}[i\lambda h_1(\lambda;\tau) \\
& + e^{-i\lambda} i\lambda h_0(\lambda;\tau) + e^{-i\lambda}\{i\lambda(e^{i\lambda}h_1(\lambda;\tau) - h_0(\lambda;\tau))\}]d\lambda \\
= -\int_{\partial D^-} & e^{-\lambda^2 t} 2i\lambda h_1(\lambda;\tau)d\lambda \\
= -\int_{\partial D^-} & e^{-\lambda^2 t} 2i\lambda \int_0^T e^{\lambda^2 s} g_1(s)dsd\lambda
\end{aligned}$$

Change variables once more: let $\lambda^2 = -i\rho \implies \lambda = -i\sqrt{i\rho}$. This produces a constructed Inverse Fourier Transform and notice that $\frac{d\rho}{d\lambda} = 2i\lambda$. Thus we see that

$$-\int_{-\mathbb{R}} e^{i\rho t} \int_0^T e^{-i\rho s} g_1 ds d\rho = 2\pi g_1(s)$$

We have demonstrated that the solution representation satisfies the boundary conditions. In addition to this, we also showed that it satisfied the PDE and the IC as well. In conclusion, the solution representation solves the problem defined and we have successfully employed the Unified Transform Method to solve the Heat Equation.

Appendix B

LS Equation Appendix

B.1 Defining the Problem

We are interested in applying the UTM to analyse the Time-Dependent, Zero Potential Linear Schrödinger Equation, which is defined by the following Partial Differential Equation (PDE), an Initial Condition (IC), and two Boundary Conditions (BC(1) and BC(2)).

$$[\partial_t + i\partial_{xx}]q(x,t) = 0 \quad \text{(PDE)}$$

$$q(x,0) = q_0(x) \quad \text{(IC)}$$

$$q(0,t) = g_0 \quad \text{(BC (A))}$$

$$q(1,t) = g_1 \quad \text{(BC (B))}$$

B.1.1 Linear System

For now, we shall take $e^{\lambda^2\tau}\hat{q}(\lambda;\tau)$ to be "known" until we deal with it momentarily. So we now have two unknowns expressed in terms of known data. Note that in (**??**), $f_j(\lambda; X, \tau)$ depends on λ entirely through λ^2 in $e^{\lambda^2\tau}$. If we apply the identity mapping $\lambda \mapsto \lambda$ and the mapping $\lambda \mapsto -\lambda$ to the Global Relation, $(GR)\Big|_{\lambda\mapsto\lambda}$ and $(GR)\Big|_{\lambda\mapsto-\lambda}$ yield a system of two linearly independent equations with the two unknowns as below.

$$-i\begin{pmatrix} -e^{-i\lambda} & 1 \\ -e^{i\lambda} & 1 \end{pmatrix}\begin{pmatrix} f_1(\lambda;1,\tau) \\ f_1(\lambda;0,\tau) \end{pmatrix} = \begin{pmatrix} M(\lambda) \\ M(-\lambda) \end{pmatrix} + \begin{pmatrix} \hat{q}_0(\lambda) \\ \hat{q}_0(-\lambda) \end{pmatrix} - e^{-i\lambda^2\tau}\begin{pmatrix} \hat{q}(\lambda;\tau) \\ \hat{q}(-\lambda;\tau) \end{pmatrix}$$

Where $M(\lambda) = -\lambda h_0(\lambda;\tau) + \lambda e^{-i\lambda} h_1(\lambda;\tau)$ and $M(-\lambda) = \lambda h_0(-\lambda;\tau) - \lambda e^{i\lambda} h_1(-\lambda;\tau)$ We solve this system using Cramer's Rule

$$\Delta(\lambda) = \begin{vmatrix} -e^{-i\lambda} & 1 \\ -e^{i\lambda} & 1 \end{vmatrix} = -e^{-i\lambda} + e^{i\lambda}$$

$$= 2isin(\lambda) \tag{$\Delta(\lambda)$}$$

$$\zeta^+(\lambda;\phi) = \begin{vmatrix} \phi(\lambda) & 1 \\ \phi(-\lambda) & 1 \end{vmatrix} = -\phi(\lambda) - \phi(-\lambda)$$

$$\zeta^-(\lambda;\phi) = \begin{vmatrix} -e^{-i\lambda} & \phi(\lambda) \\ -e^{i\lambda} & \phi(-\lambda) \end{vmatrix} = -\phi(-\lambda)e^{-i\lambda} + \phi(\lambda)e^{i\lambda}$$

Using (**??**) and (**??**), we recast the unknowns in (**??**) as follows

$$f_1(\lambda;0,\tau) = \frac{\zeta^-(\lambda;M(\cdot;\tau)}{\Delta(\lambda))} + \frac{\zeta^-(\lambda;\hat{q}_0(\lambda))}{\Delta(\lambda)} - e^{-i\lambda^2\tau}\left(\frac{\zeta^-(\lambda;\hat{q}(\cdot;\tau))}{\Delta(\lambda)}\right)$$

and

$$f_1(\lambda;1,\tau) = \frac{\zeta^+(\lambda;M(\cdot;\tau)}{\Delta(\lambda))} + \frac{\zeta^+(\lambda;\hat{q}_0(\lambda))}{\Delta(\lambda)} - e^{-i\lambda^2\tau}\left(\frac{\zeta^+(\lambda;\hat{q}(\cdot;\tau))}{\Delta(\lambda)}\right)$$

B.1.2 Perturbing Around Poles

From the above, we see that the zeroes of $\Delta(\lambda)$ are $n\pi$. Before proceeding, we must deform $\partial D^{\pm}$ around these zeroes. To achieve this, we use Cauchy's Theorem in conjunction with a reconstruction of $\partial D^{\pm}$ with a simple, closed curve Γ as follows

Definition B.1.1 *(Parametrisation of* Γ*)*

Let Γ *be a closed unit circle oriented clockwise around every pole of* $\Delta(\lambda)$*:*

$$x = n\pi + sin\theta$$

$$y = cos\theta$$

for $\theta \in [\pi, 2\pi), n \in [0, +\infty)$

Let $\partial D^{\pm} = \gamma$. Then, with the parametrisation of Γ above, we are able to use Cauchy's Theorem to demonstrate that

$$\int_{\gamma} f(z)dz = \int_{\Gamma'} f(z)dz + \int_{\Gamma} f(z)dz$$

Where $\int_{\Gamma} f(z)dz = 0$ by Cauchy's Theorem We obtain $\widetilde{\partial D^{\pm}}$, which repre-

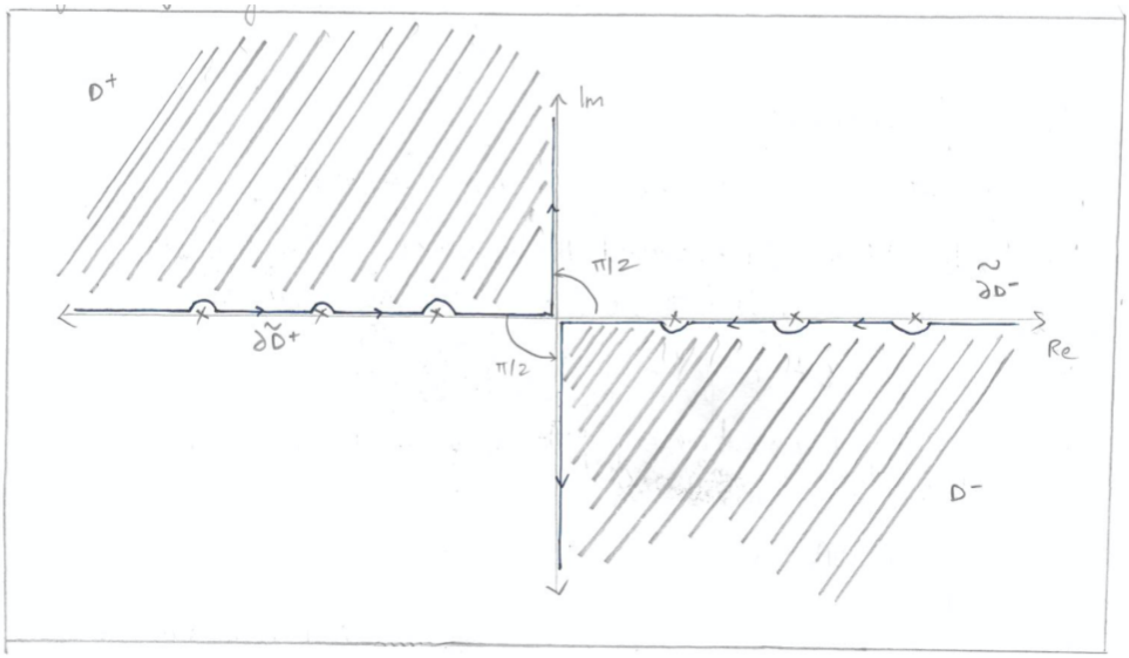

FIGURE B.1: Linear Schrödinger Adjusted Contour Sectors

sents the perturbed $\partial D^{\pm}$ using the method above. So, EF_{τ} becomes

$$\begin{aligned}
2\pi q(x,t) &= \int_{-\infty}^{\infty} e^{i\lambda x + i\lambda^2 t}\hat{q}(\lambda)d\lambda \\
&+ \int_{\partial\widetilde{D^-}} e^{i\lambda(x-1)+i\lambda^2 t}(if_1(\lambda;1;\tau) - \lambda f_0(\lambda;1;\tau))d\lambda \\
&+ \int_{\partial\widetilde{D^+}} e^{i\lambda x + i\lambda^2 t}(if_1(\lambda;0;\tau) - \lambda f_0(\lambda;0;\tau))d\lambda \qquad (\widetilde{EF_{\tau}}) \\
&\text{valid for } (x,t) \in (0,1) \times [0,\tau], \tau \in [0,T]
\end{aligned}$$

B.1.3 Solution Representation

Now we can safely move on. We have expressed our hitherto unknown data in terms of known data. But not entirely, since we have been pretending that $\hat{q}(\cdot;\tau)$ is data when it is not. It is now time to deal with it. Substituting the above into EF_{τ} and taking the remaining non-data integrands outside gives us

$$
\begin{aligned}
2\pi q(x,t) = &\int_{-\infty}^{\infty} e^{i\lambda x - i\lambda^2 t}\hat{q}(\lambda)d\lambda + \int_{\partial\widetilde{D^-}} \text{data } d\lambda + \int_{\partial\widetilde{D^+}} \text{data } d\lambda \\
&- i\int_{\partial\widetilde{D^-}} e^{i\lambda(x-1)}e^{i\lambda^2(t-\tau)}\left(\frac{\zeta^+(\lambda;\hat{q}(\cdot;\tau))}{\Delta(\lambda)}\right)d\lambda \\
&- i\int_{\partial\widetilde{D^+}} e^{i\lambda x}e^{i\lambda^2(t-\tau)}\left(\frac{\zeta^-(\lambda;\hat{q}(\cdot;\tau))}{\Delta(\lambda)}\right)d\lambda
\end{aligned}
$$

Our aim is to show the integrals involving $\hat{q}(\cdot;\tau)$ evaluate to 0. We do this by employing Jordan's Lemma. We see that the ratio term in the integrands we are interested in is $\mathcal{O}(|\lambda|^{-1})$, uniformly in $arg(\lambda)$ as $\lambda \to \infty$ within $Clos(D^{\pm})$. Hence, both of these integrals evaluate to zero within their respective closures. Having done this, we have arrived at the Solution Representation for the Heat Equation in terms of contour integrals around D

$$
\begin{aligned}
2\pi q(x,t) = &\int_{-\infty}^{\infty} e^{i\lambda x + i\lambda^2 t}\hat{q}(\lambda)d\lambda \\
&+ \int_{\partial\widetilde{D^-}} ie^{i\lambda(x-1)+i\lambda^2 t}\left((-\lambda h_1(\lambda;\tau)) + \frac{\zeta^+(\lambda; M(\cdot;\tau) + \hat{q_0}}{\Delta(\lambda))}\right)d\lambda \\
&+ \int_{\partial\widetilde{D^+}} ie^{i\lambda x+i\lambda^2 t}\left((-\lambda h_0(\lambda;\tau)) + \frac{\zeta^-(\lambda; M(\cdot;\tau) + \hat{q_0}}{\Delta(\lambda))}\right)d\lambda
\end{aligned} \tag{SRT_τ}
$$

Where h_j, M, $\hat{q_0}$, $\zeta^{\pm}$ are explicitly defined in the problem data.

B.2 Stage III

Having derived a Solution Representation, we define $q(x,t)$ using (SRT_τ) and verify if it indeed solves the problem we defined in Stage I by checking if it satisfies the PDE, IC, and BCs.

B.2.1 PDE

Any $(x,t) \in (0,1) \times (0,T)$ has a closed neighbourhood Ω within $(0,1) \times (0,T)$ such that $e^{i\lambda x' + i\lambda^2 t'} \to 0$ exponentially uniformly on $(x', t') \in \Omega$ as $\lambda \to \infty$ along $\mathbb{R}$ or $\partial D^\pm$

Therefore, all partial derivatives of q exist and are given by differentiating the integrand. Taking ∂ as defined in (SRT_τ) above and integrating therefore does satisfy the PDE.

B.2.2 Initial Condition

We aim to isolate the initial conditions from our (SRT_τ). Before we accomplish this, note that $\forall T \in (SR_T)$, T can be replaced by τ, $\forall \tau \in [t,T]$. So, q is equivalently defined by both (SR_T) and ((SRT_τ)). When we set $\tau = t$ and $t = 0$, $h_j(\lambda;0) = 0 \implies M(\lambda;0) = 0$. Thus we cancel these terms and are left with a Solution Representation in terms of $\hat{q}_0$

$$2\pi q(x,0) = \int_{-\infty}^{\infty} e^{i\lambda x} \hat{q}(\lambda) d\lambda + \underbrace{\int_{\widetilde{\partial D^+}} e^{i\lambda x} \left(\frac{\zeta^-(\lambda;\hat{q}_0)}{\Delta(\lambda)} \right) d\lambda + \int_{\widetilde{\partial D^-}} e^{i\lambda(x-1)} \left(\frac{\zeta^+(\lambda;\hat{q}_0)}{\Delta(\lambda)} \right) d\lambda}_{=0 \text{ by Jordan's Lemma}}$$

Thus, we reduce the equation above to an Inverse Fourier Transform

$$\implies 2\pi q(x,0) = q_0(x), \forall x \in (0,1)$$

We have thus shown that (SRT_τ) does satisfy (IC).

B.2.3 Boundary Conditions

Similar to our work with the Initial Condition above, we aim to isolate (BC (A)) and (BC (B)) within our derived (SRT_τ). To do this, we need to reconstruct SR_τ as a series representation.

Series Representation

Recall our adjusted ($\widetilde{EF_\tau}$). We first use Jordan's Lemma to deform the (undeformed in ($\widetilde{EF_\tau}$)) part of $\int_{\partial D^\pm}$ dependent on $\hat{q}_0$ to the real line perturbed along semicircular contours of radius ϵ around the poles of ($\Delta(\lambda)$), ($k\pi, k \in \mathbb{Z}$). That is, we deform the positive and negative imaginary axes to the real line. We also deform $\int_{-\infty}^{\infty} e^{i\lambda x}\hat{q}_0(\lambda)d\lambda$ "up" around each zero of ($\Delta(\lambda)$), to $\Gamma^\pm$. Observe that

$$\hat{q}_0(\lambda) = \frac{ie^{-i\lambda}\zeta^+(\lambda;\hat{q}_0) - i\zeta^-(\lambda;\hat{q}_0)}{\Delta(\lambda)} \tag{B.1}$$

We use (B.1) above in conjunction with the deformations described above and move terms dependent on $\hat{q}_0$ from the latter two integrals to the first integral. This yields

$$\begin{aligned}2\pi q(x,t) = \sum_{k\in\mathbb{Z}} \int_{C(k\pi,\epsilon)} e^{i\lambda x+i\lambda^2 t} i \left(\frac{\hat{q}_0(-\lambda)e^{-i\lambda} - \hat{q}_0(\lambda)e^{i\lambda}}{e^{i\lambda}-e^{-i\lambda}}\right) d\lambda \\ + \int_{\partial\widetilde{D^-}} e^{i\lambda(x-1)+i\lambda^2 t} \left(\frac{\zeta^+(\lambda; M(\cdot;\tau))}{\Delta(\lambda)} - (\lambda h_1(\lambda;\tau)\right) d\lambda \\ + \int_{\partial\widetilde{D^+}} e^{i\lambda x+i\lambda^2 t} \left(\frac{\zeta^-(\lambda; M(\cdot;\tau))}{\Delta(\lambda)} - (\lambda h_0(\lambda;\tau)\right) d\lambda\end{aligned} \qquad (SRT_\tau - Series)$$

Homogeneous Boundary Conditions

Studying ($SRT_\tau - Series$), we notice two things of importance: first, regardless if $x = 0$ or $x = 1$, residues cancel out since $\int_{C(k\pi,)} \ldots d\lambda = -\int_{C(-k\pi,)} \ldots d\lambda$. This is akin to eliminating the first integral as a self-cancelling sum of circular integrals around poles that mirror each other (for every $k\pi$, there is an equivalent $-k\pi$). Taking this fact together with the second and third integrals evaluating to 0 if (BC (A)) and (BC (B)) are homogeneous, we can conclude that (SRT_τ) satisfies the Boundary Conditions for the Heat Equation provided they are homogeneous.

Inhomogeneous Boundary Conditions

If (BC (A)) and (BC (B)) are inhomogeneous, the fact remains that regardless if $x = 0$ or $x = 1$, residues cancel out since $\int_{C(k\pi,)} \ldots d\lambda = -\int_{C(-k\pi,)} \ldots d\lambda$. Now, for each case $x = 0$ and $x = 1$, we aim to show that the remainder of ($SRT_\tau - Series$) yields the Boundary Conditions. We begin with the case where $x = 0$. We change variables in $\int_{\partial\widetilde{D^-}} \ldots d\lambda$

such that $\lambda \mapsto -\lambda$. We combine both integrals and simplify to see that

$$
\begin{aligned}
2\pi q(0,t) &= \int_{\partial\widetilde{D^+}} ie^{i\lambda^2 t}[-\lambda h_0(\lambda;\tau) + e^{i\lambda}\lambda h_1(\lambda;\tau) - \lambda h_0(\lambda;\tau) + \lambda e^{i\lambda} h_1(\lambda;\tau)]d\lambda \\
&= -\int_{\partial\widetilde{D^+}} ie^{i\lambda^2 t} 2\lambda e^{i\lambda} h_0(\lambda;\tau) d\lambda \\
&= -\int_{\partial\widetilde{D^+}} e^{i\lambda^2 t} 2i\lambda e^{i\lambda} \int_0^T e^{-i\lambda^2 s} g_0(s) \quad dsd\lambda
\end{aligned}
$$

Change variables to find some $\lambda^2 = r_1$ such that this r_1 produces a constructed inverse Fourier transform.

Then check the case where $x = 1$. We change variables in $\int_{\partial\widetilde{D^+}} \dots d\lambda$ such that $\lambda \mapsto -\lambda$. We combine both integrals and simplify to see that

$$
\begin{aligned}
2\pi q(1,t) &= \int_{\partial\widetilde{D^-}} ie^{i\lambda^2 t}[-\lambda h_1(\lambda;\tau) + e^{i\lambda}\lambda h_0(\lambda;\tau) + e^{i\lambda}(-\lambda h_0(\lambda;\tau) - e^{i\lambda}\lambda h_1(\lambda;\tau))]d\lambda \\
&= -\int_{\partial\widetilde{D^-}} ie^{i\lambda^2 t} h_1(\lambda;\tau)(\lambda - e^{2i\lambda}) d\lambda \\
&= -\int_{\partial\widetilde{D^-}} e^{i\lambda^2 t}(\lambda - e^{2i\lambda}) \int_0^T e^{-i\lambda^2 s} g_1(s) \quad dsd\lambda
\end{aligned}
$$

Change variables to find some $\lambda^2 = r_2$ such that this r_2 produces a constructed inverse Fourier transform.

Appendix C

LKdV Equation Appendix

C.1 Defining the Problem

We are interested in applying the UTM to analyse the Linearised Korteweg-De Vries Equation, which is defined by the following Partial Differential Equation (PDE), an Initial Condition (IC), and three Boundary Conditions (BC(1), BC(2), and BC(3)).

$$[\partial_t - \partial_{xxx}]q(x,t) = 0 \tag{PDE}$$

$$q(x,0) = q_0(x) \tag{IC}$$

$$q(0,t) = g_0 \tag{BC (1)}$$

$$q(1,t) = g_1 \tag{BC (2)}$$

$$q_x(0,t) = g_2 \tag{BC (3)}$$

C.1.1 Linear System

For now, we shall take $e^{i\lambda^3\tau}\hat{q}(\lambda;\tau)$ to be "known" until we deal with it momentarily. So we now have three unknowns expressed in terms of known data. Note that in (**??**), $f_j(\lambda; X, \tau)$ depends on λ entirely through λ^3 in $e^{i\lambda^3\tau}$. If we apply the mappings $\lambda \mapsto \lambda$, $\lambda \mapsto \alpha\lambda$, and $\lambda \mapsto \alpha^2\lambda$ ($\alpha \in \mathbb{C} = \sqrt[3]{1}$) to the Global Relation, $(GR)\Big|_{\lambda\mapsto\lambda}$, $(GR)\Big|_{\lambda\mapsto\alpha\lambda}$, and $(GR)\Big|_{\lambda\mapsto\alpha^2\lambda}$ yield a system of three linearly independent equations with the three unknowns as below.

$$\begin{pmatrix} 1 & -e^{-i\lambda} & -i\lambda e^{-i\lambda} \\ 1 & -e^{i\alpha\lambda} & -i\alpha\lambda e^{-i\alpha\lambda} \\ 1 & -e^{i\alpha^2\lambda} & -i\alpha^2\lambda e^{-i\alpha^2\lambda} \end{pmatrix} \begin{pmatrix} f_2(\lambda; 0, \tau) \\ f_2(\lambda; 1, \tau) \\ f_1(\lambda; 1, \tau) \end{pmatrix} = \begin{pmatrix} M(\lambda) \\ M(\alpha\lambda) \\ M(\alpha^2\lambda) \end{pmatrix} + \begin{pmatrix} \hat{q}_0(\lambda) \\ \hat{q}_0(\alpha\lambda) \\ \hat{q}_0(\alpha^2\lambda) \end{pmatrix} - e^{i\lambda^3\tau} \begin{pmatrix} \hat{q}(\lambda; \tau) \\ \hat{q}(\alpha\lambda; \tau) \\ \hat{q}(\alpha^2\lambda; \tau) \end{pmatrix}$$

Where $M(\lambda) = -e^{-i\lambda}\lambda^2 h_1(\lambda;\tau) + \lambda^2 h_0(\lambda;\tau) - i\lambda h_2(\lambda;\tau)$ We solve this system using Cramer's Rule

$$\Delta(\lambda) = \begin{vmatrix} 1 & -e^{-i\lambda} & -i\lambda e^{-i\lambda} \\ 1 & -e^{i\alpha\lambda} & -i\alpha\lambda e^{-i\alpha\lambda} \\ 1 & -e^{i\alpha^2\lambda} & -i\alpha^2\lambda e^{-i\alpha^2\lambda} \end{vmatrix}$$

$$= -i\lambda(e^{-i\lambda(1+\alpha)}(1-\lambda) + e^{-i\alpha\lambda(1+\alpha)}(\alpha-\alpha^2)$$
$$+ e^{-i\lambda(1+\alpha^2)}(\alpha^2-1) \qquad (\Delta(\lambda))$$

$$\zeta'(\lambda;\phi) = \begin{vmatrix} \phi(\lambda) & -e^{-i\lambda} & -i\lambda e^{-i\lambda} \\ \phi(\alpha\lambda) & -e^{i\alpha\lambda} & -i\alpha\lambda e^{-i\alpha\lambda} \\ \phi(\alpha^2\lambda) & -e^{i\alpha^2\lambda} & -i\alpha^2\lambda e^{-i\alpha^2\lambda} \end{vmatrix}$$

$$= \phi(\lambda)(e^{-i\alpha\lambda(1+\alpha)}(i\alpha\lambda(\alpha-1))$$
$$+ i\alpha\lambda(\phi(\alpha^2\lambda)(e^{-i\lambda(1+\alpha)} - \alpha(\phi(\alpha\lambda)(e^{-i\lambda(1+\alpha^2)}))$$
$$- i\lambda(\phi(\alpha^2\lambda)(e^{-i\lambda(1+\alpha)}) - \phi(\alpha\lambda)(e^{-i\lambda(1+\alpha^2)})) \qquad (C.1)$$

$$\zeta''(\lambda;\phi) = \begin{vmatrix} 1 & \phi(\lambda) & -i\lambda e^{-i\lambda} \\ 1 & \phi(\alpha\lambda) & -i\alpha\lambda e^{-i\alpha\lambda} \\ 1 & \phi(\alpha^2\lambda) & -i\alpha^2\lambda e^{-i\alpha^2\lambda} \end{vmatrix}$$

$$= -i\lambda(\phi(\lambda)(\alpha(e^{-i\alpha\lambda} - \alpha e^{-i\alpha^\lambda}))$$
$$+ \phi(\alpha\lambda)(\alpha^2 e^{-i\alpha^2\lambda} - e^{-i\lambda})$$
$$+ \phi(\alpha^2\lambda)(e^{-i\lambda} - \alpha e^{-i\alpha\lambda})) \qquad (C.2)$$

$$\zeta'''(\lambda;\phi) = \begin{vmatrix} 1 & -e^{-i\lambda} & \phi(\lambda) \\ 1 & -e^{i\alpha\lambda} & \phi(\alpha\lambda) \\ 1 & -e^{i\alpha^2\lambda} & \phi(\alpha^2\lambda) \end{vmatrix}$$

$$= \phi(\lambda)(e^{i\alpha\lambda} - e^{i\alpha^2\lambda})$$
$$+ \phi(\alpha\lambda)(^{i\alpha^2\lambda} - e^{-i\lambda})$$
$$+ \phi(\alpha^2\lambda)(e^{-i\lambda} - e^{i\alpha\lambda}) \qquad (C.3)$$

We take (C.1), (C.2), and (C.3) and recast the unknowns in (**??**) as follows

$$f_2(\lambda;0,\tau) = \frac{\zeta'(\lambda;M(\cdot;\tau)}{\Delta(\lambda)}) + \frac{\zeta'(\lambda;\hat{q_0}(\lambda))}{\Delta(\lambda)} - e^{i\lambda^3\tau}\left(\frac{\zeta'(\lambda;\hat{q}(\cdot;\tau))}{\Delta(\lambda)}\right)$$

and

$$f_2(\lambda;1,\tau) = \frac{\zeta''(\lambda;M(\cdot;\tau)}{\Delta(\lambda)}) + \frac{\zeta''(\lambda;\hat{q_0}(\lambda))}{\Delta(\lambda)} - e^{i\lambda^3\tau}\left(\frac{\zeta''(\lambda;\hat{q}(\cdot;\tau))}{\Delta(\lambda)}\right)$$

and

$$f_1(\lambda;1,\tau) = \frac{\zeta'''(\lambda;M(\cdot;\tau)}{\Delta(\lambda)}) + \frac{\zeta'''(\lambda;\hat{q_0}(\lambda))}{\Delta(\lambda)} - e^{i\lambda^3\tau}\left(\frac{\zeta'''(\lambda;\hat{q}(\cdot;\tau))}{\Delta(\lambda)}\right)$$

C.1.2 Solution Representation

We have expressed our hitherto unknown data in terms of known data. But not entirely, since we have been pretending that $\hat{q}(\cdot;\tau)$ is data when it is not. It is now time to deal with it. Substituting the above into EF_τ and taking the remaining non-data integrands outside gives us

$$\begin{aligned}
2\pi q(x,t) = & \int_{-\infty}^{\infty} e^{i\lambda x - i\lambda^3 t}\hat{q}(\lambda)d\lambda - \int_{\partial D^-} \text{data } d\lambda + \int_{\partial D^+} \text{data } d\lambda \\
& - \int_{\partial D^-} e^{i\lambda(x-1)}e^{i\lambda^3(\tau-t)}\left(\frac{\zeta''(\lambda;\hat{q}(\cdot;\tau))}{\Delta(\lambda)}\right)d\lambda \\
& - \int_{\partial D^-} \lambda e^{i\lambda x}e^{i\lambda^3(\tau)-t}\left(\frac{\zeta'''(\lambda;\hat{q}(\cdot;\tau))}{\Delta(\lambda)}\right)d\lambda \\
& - \int_{\partial D^+} e^{i\lambda x}e^{i\lambda^3(\tau-t)}\left(\frac{\zeta'(\lambda;\hat{q}(\cdot;\tau))}{\Delta(\lambda)}\right)d\lambda
\end{aligned}$$

Our aim is to show the integrals involving $\hat{q}(\cdot;\tau)$ evaluate to 0. We do this by employing Jordan's Lemma. We see that the ratio term in the integrands we are interested in = $\mathcal{O}(|\lambda|^{-2})$, uniformly in $arg(\lambda)$ as $\lambda \to \infty$ within $Clos(D^{\pm})$. Hence, both of these integrals evaluate to zero within their respective closures. Having done this, we have arrived at the Solution Representation for the Linearised Kortweg-De Vries Equation in terms of contour integrals around D

$$\begin{aligned}2\pi q(x,t) &= \int_{-\infty}^{\infty} e^{i\lambda x - i\lambda^3 t}\hat{q}(\lambda)d\lambda \\ &- \int_{\partial D^-} e^{i\lambda(x-1)-i\lambda^3 t}\left(i\left(\frac{\zeta'' + (\lambda; M(\cdot;\tau) + \hat{q_0})}{\Delta(\lambda)}\right)\right. \\ &- \lambda\left(\frac{\zeta''' + (\lambda; M(\cdot;\tau) + \hat{q_0})}{\Delta(\lambda)}\right) - i\lambda^2 h_1(\lambda;\tau)d\lambda \\ &+ \int_{\partial D^+} e^{i\lambda x - i\lambda^3 t}\left(\frac{\zeta'(\lambda; M(\cdot;\tau) + \hat{q_0}}{\Delta(\lambda)}) - i\lambda h_2(\lambda;\tau) + \lambda^2 h_0(\lambda;\tau)\right) d\lambda\end{aligned} \tag{SRT_τ}$$

Where h_j, M, $\hat{q_0}$, $\zeta^{\pm}$ are explicitly defined in the problem data.

C.2 Stage III

Having derived a Solution Representation, we define $q(x,t)$ using (SRT_τ) and verify if it indeed solves the problem we defined in Stage I by checking if it satisfies the PDE, IC, and BCs.

C.2.1 PDE

As a consequence of Uniform Convergence, we know that $(x,t) \in (0,1) \times (0,T)$ has a closed neighbourhood Ω within $(0,1) \times (0,T)$. On Ω, $e^{i\lambda x' - i\lambda 3t'} \to$

0 exponentially uniformly on $(x', t') \in \Omega$ as $\lambda \to \infty$ along $\mathbb{R}$ or $\partial D^{\pm}$ Therefore, all partial derivatives of q exist and are given by differentiating the integrand. Taking ∂ as defined in (SRT_τ) above and integrating therefore does satisfy the PDE.

C.2.2 Initial Condition

We aim to isolate the initial conditions from our (SRT_τ). Before we accomplish this, note that $\forall T \in (SR_T)$, T can be replaced by τ, $\forall \tau \in [t, T]$. So, q is equivalently defined by both (SR_T) and ((SRT_τ)). When we set $\tau = t$ and $t = 0$, $h_j(\lambda; 0) = 0 \implies M(\lambda; 0) = 0$. Thus we cancel these terms and are left with a Solution Representation in terms of $\hat{q}_0$

$$
\begin{aligned}
2\pi q(x,0) = & \int_{-\infty}^{\infty} e^{i\lambda x}\hat{q}(\lambda) d\lambda \\
& - \int_{\partial D^-} e^{i\lambda(x-1)} \left\{ i \left(\frac{\zeta''(\lambda; \hat{q_0})}{\Delta(\lambda)} \right) - \lambda \left(\frac{\zeta'''(\lambda; \hat{q_0})}{\Delta(\lambda)} \right) \right\} d\lambda \\
& + \int_{\partial D^+} e^{i\lambda x} \left(\frac{\zeta'(\lambda; \hat{q_0})}{\Delta(\lambda)} \right) d\lambda
\end{aligned}
$$

Where the latter two integrals $= 0$ by Jordan's Lemma.Thus, we reduce the equation above to an Inverse Fourier Transform

$$
\implies 2\pi q(x,0) = q_0(x), \forall x \in (0,1)
$$

We have thus shown that (SRT_τ) does satisfy (IC).