SRP Reports

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July 2021

Chapter 1

Jordan Normal Form

Jordan Normal Form is a matrix representation which exists for all matrices, and is almost diagonalized. It has the eigenvalues of the matrix along the diagonal as well as other 1s populated along the superdiagonal.

1.1 Jordan Normal Form

For a given matrix $A, \exists J$ such that A is similar to J using an invertible matrix S, given by the relation:

$$A = SJS^{-1}$$

It is the direct sum ($\oplus$) of several Jordan Blocks ($J_{d,\lambda}$) formed using Jordan Chains.

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

$$J_{d,\lambda} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

where d is the dimension of the square Jordan Block.

$$J = \bigoplus_{n,i} J_{d_n,\lambda_i}, n \in \{1, \ldots, k\}$$

where k is the number of Jordan Chains for the i-th eigenvalue of A.

1.2 Diagonalizability

A matrix is diagonalizable if its algebraic multiplicity is equal to its geometric multiplicity for all eigenvalues. This is equivalent to saying that there exists a basis of the space consisting of the eigenvectors of the matrix.

All matrices are not diagonalizable as seen by the Spectral Theorem. Therefore, the Jordan Normal Form is obtained using a basis of generalized eigenvectors of the matrix.

1.3 Generalized Eigenvectors

A generalized eigenvector v of Rank k has the form

$$(A - \lambda I)^k v = 0$$
$$(A - \lambda I)^i v \neq 0, i < k \in \mathbb{N}.$$

A generalized eigenvector of Rank 1 is an eigenvector.

1.4 Jordan Chain

For a generalized eigenvector of rank k, v_k, a Jordan Chain $C(v_k)$ is the set of generalized eigenvectors,

$$C(v_k) = \{v_k, (A - \lambda I)v_k, (A - \lambda I)^2 v_k, \ldots, (A - \lambda I)^{k-2} v_k, (A - \lambda I)^{k-1} v_k\}$$

which can be rewritten as

$$C(v_k) = \{v_k, v_{k-1}, v_{k-2}, \ldots, v_2, v_1\}$$

where $v_j = (A - \lambda I)^{k-j} v_k = (A - \lambda I)v_{j+1}$ for $j \in \{1, \ldots, k-1\}$.

The final element of this chain is an eigenvector and using the recurrence relation mentioned above, starting from the eigenvector we can find generalized eigenvectors or $v_j + 1$ starting from v_j which satisfies the relation. This terminates when there is no solution to the relation and then the Jordan Chain is complete.

1.5 Determining the Jordan Normal Form

- **Determine the Algebraic Multiplicity**

  Using either the characteristic polynomial or by determining the eigenvalues, note the algebraic multiplicity for each eigenvalue.

- **Determine the Geometric Multiplicity**

  The number of linearly independent eigenvectors corresponding to the eigenvalue gives the geometric multiplicity of the eigenvalues which tells us the number of Jordan Blocks for that eigenvalue.

- **Determine the Length of the Jordan Chains**

  The length of the Jordan Chain gives us information about the dimension of the Jordan Block to which it corresponds. Note down the dimensions of the null space of $(A-\lambda I)^p$ starting from $p = 1$ and iteratively increasing p until the dimension of the null space is equal to the algebraic multiplicity for that eigenvalue. This gives information about the length of the Jordan Chains.

  Example: Algebraic Multiplicity = 4,

  $$\dim(A - \lambda I) = 2, \dim(A - \lambda I)^2 = 3, \dim(A - \lambda I)^3 = 4,$$

  tells us that there are 2 Jordan Chains, of length 1 and 3.

- **Determining the Similarity Matrix**

  The similarity matrix, is a basis of the generalized eigenvectors of the original matrix. Let the length of the longest Jordan Chain be k, so iterate from $p = 1$ until k at each step determining the basis of the null space of $(A - \lambda I)^p$. When increasing p make sure to add vectors to the basis for the lower dimension. When p reaches k, we get all the generalized eigenvectors for the matrix and the only thing left to do is arrange them into the matrix.

  To arrange the eigenvectors for a particular Jordan Block, start with the lowest degree generalized eigenvector and then add the higher degree generalized eigenvectors forming that Jordan Chain to the list. Then do the next generalized eigenvector and add the other generalized eigenvectors which form the Jordan Chain, and so on the entire basis is arranged. This is our similarity matrix S.

Chapter 2

2x2 Non-Diagonalizable Proof

2.1 Introduction

This proof will show that for a 2×2 system, the only matrix polynomial possible is when the eigenvalue is a polynomial in k (or λ depending on source) itself.

2.2 Same Eigenvalue

It is apparent that the eigenvalues of a non-diagonalizable 2×2 matrix must because if there are 2 Jordan Blocks of dimension 1, the matrix would be diagonalizable which is against our assumption for the proof.

Thus, there is one Jordan Block of dimension 1, and since all diagonal elements of a Jordan Block have the same eigenvalue, the Matrix Polynomial to be determined must have the same eigenvalue.

2.3 Proof

Let Λ be a matrix polynomial in k, P be the similarity transformation and J be the matrix expressed in Jordan Normal Form, such that

$$\Lambda = PJP^{-1}.$$

The right hand side can be represented as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Omega & 1 \\ 0 & \Omega \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}.$$

Multiplying the matrices gives us a general expression for Λ as

$$\Lambda = \begin{pmatrix} \Omega - \frac{ac}{ad-bc} & a^2 \\ -c^2 & \Omega + \frac{ac}{ad-bc} \end{pmatrix}.$$

However, since Λ is also a matrix polynomial, we must have that

$$\Lambda = \sum_{i=0}^{n} k^i A_i$$

where A_i refers to a matrix with complex entries.

Substituting X for $\frac{ac}{ad-bc}$, equating the two expressions and considering the diagonal entries only, we get the following system of equations.

$$\Omega - X = \sum_{i=0}^{n} \alpha_i k^i$$

$$\Omega + X = \sum_{i=0}^{n} \beta_i k^i.$$

Solving for Ω, we get that

$$\Omega = \sum_{i=0}^{n} \frac{(\alpha_i - \beta_i)}{2} k^i.$$

2.4 Conclusion

From this, we learn that using the Unified Transform Method for a 2×2 System of PDEs with a non-diagonalizable matrix-valued polynomial reduces to the general case when its eigenvalue is a polynomial itself.

Chapter 3

UTM for Systems of Non-Diagonalizable Differential Operators

3.1 Introduction

This paper focuses on extending the Unified Transform Method for Systems of Linear Equations as presented by Deconinck et al. to the case when the differential operator matrix is non-diagonalizable. It has been tested only for the 2×2 system case where the eigenvalue must itself be a polynomial.

3.2 System of PDEs

The system of PDEs is given by the following equations. Seek continuous functions p, q valid over the half line which satisfy

$$q_t + q_{xx} = 0 \tag{3.1}$$

$$p_t + p_{xx} + q = 0. \tag{3.2}$$

The system of equations follow the general form of

$$Q_t + \Lambda(-i\partial_x)Q = 0 \tag{3.3}$$

which in this case is,

$$\partial_t \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} \partial_x^2 & 0 \\ 1 & \partial_x^2 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = 0 \tag{3.4}$$

Note that $\Lambda(-i\partial_x)$ is non-diagonalizable.

3.3 Setting up the Global Relation

Substitute

$$Q = \begin{pmatrix} Q_1 \\ \vdots \\ Q_n \end{pmatrix} e^{ikx-\omega t} \tag{3.5}$$

in (3.3), where $k \in \mathbb{R}$, so that we get ω which satisfies

$$\det(\Lambda(k) - \omega I) = 0. \tag{3.6}$$

In our example, the dispersion relation is $\omega = k^2$. Furthermore, it is of importance to note that in this case there are no dispersion branches there is a 0 on the off-diagonal of this 2×2 matrix valued-polynomial. The implication of this is that there are no branch points for the given example, which may not always be the case.

The local relation is obtained using Lax Pair Formulation and has the following structure,

$$(e^{-ikxI+\Lambda(k)t}Q)_t - (e^{-ikxI+\Lambda(k)t}X(x,t,k)Q)_x = 0 \tag{3.7}$$

$$(e^{-ikxI+J(k)t}P(k)Q)_t - (e^{-ikxI+J(k)t}P(k)X(x,t,k)Q)_x = 0 \tag{3.8}$$

where $J(k)$ is in Jordan Normal Form and is similar to $\Lambda(k)$ as given by,

$$\Lambda(k) = P^{-1}(k)J(k)P(k) \tag{3.9}$$

and $X(x,t,k)$ is a differential operator of degree at most $n-1$, polynomial in k, given by

$$X(x,t,k) = i \left. \frac{\Lambda(k) - \Lambda(l)}{k-l} \right|_{l=i\partial_x} = \sum_{j=0}^{n-1} c_j(k)\partial_x^j. \tag{3.10}$$

3.4 Exponentiating Non-Diagonalizable Matrices

In the previous section, we see the term, $e^{J(k)}$. In order to solve for this term, we need to use the fact that a matrix in Jordan Normal Form is the sum of a Diagonal Matrix D and a Nilpotent matrix M, a null matrix with either 1 or 0 as the elements of the superdiagonal.

$$J(k) = D(k) + M \tag{3.11}$$

Exponentiating this matrix reduces to the product of the exponents of D and M respectively.

Exponentiating a diagonal matrix reduces to exponentiating its entries.

Exponentiating a Nilpotent matrix is reduced to applying it to the Maclaurin Series for the exponential function given by:

$$e^M = \sum_{i=0}^{\infty} \frac{M^i}{i!}. \tag{3.12}$$

For a nilpotent matrix M, there exists a finite p such that

$$M^p = 0, \tag{3.13}$$

which means that the exponential is a finite computation.

Ultimately,

$$e^{J(k)} = e^{D(k)} e^M. \tag{3.14}$$

3.5 Local Relation

Using (3.8), we see that our local relations for the given example are:

$$(e^{-ikx+k^2t}(q+p))_t - (e^{-ikx+k^2t}(ik(q+p)+(q_x+p_x)))_x = 0 \tag{3.15}$$

$$(e^{-ikx+k^2t}q)_t - (e^{-ikx+k^2t}(ikq+q_x))_x = 0. \tag{3.16}$$

Note that one equation is part of the other and substituting its value as 0 gives us an equation identical to (3.16) with p in all places except for q. However, we still choose to continue using (3.15) and (3.16).

Some of the matrices used for this are

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} ikq + q_x \\ ikp + p_x \end{pmatrix} \tag{3.17}$$

and

$$e^{J(k)} = \begin{pmatrix} e^{k^2t} & e^{k^2t} \\ 0 & e^{k^2t} \end{pmatrix}. \tag{3.18}$$

3.6 Global Relation

We integrate with respect to x and s, where t has been renamed to s along the half line and from 0 to t respectively.

Integrating (3.7), (3.8) as explained above give you the following equations respectively

$$\hat{Q}_0(k) - e^{\Lambda(k)t}\hat{Q}(k,t) - G(k,t) = 0 \tag{3.19}$$

$$P(k)\hat{Q}_0(k) - e^{J(k)t}P(k)\hat{Q}(k,t) - \tilde{G}(k,t) = 0 \tag{3.20}$$

where

$$\hat{Q}_0(k) = \int_0^\infty e^{-ikx} Q_0(x)dx, \qquad \hat{Q}(k,t) = \int_0^\infty e^{-ikx} Q(x,t)dx, \tag{3.21}$$

$$G(k,t) = \int_0^t e^{\Lambda(k)s} X(0,s,k)Q(0,s)ds, \tag{3.22}$$

$$\tilde{G}(k,t) \int_0^t e^{J(k)s} P(k)X(0,s,k)Q(0,s)ds. \tag{3.23}$$

The Global Relations for the example are taken solved for using (3.15) and (3.16), and are the following:

$$e^{k^2t}(\hat{q}(x,t) + \hat{p}(x,t)) - (\hat{q}_0 + \hat{p}_0) - ik(f_0(q,0,t,k) + f_0(p,0,t,k)) \\ - (f_1(q,0,t,k) + f_1(p,0,t,k)) = 0 \tag{3.24}$$

$$e^{k^2t}\hat{q}(x,t) - \hat{q}_0 - ikf_0(q,0,t,k) - f_1(q,0,t,k) = 0 \tag{3.25}$$

where

$$f_j(\rho, X, T, k) = \int_0^T e^{k^2 s} \partial_x^j \rho(X,s)ds. \tag{3.26}$$

3.7 Ehrenpreis Form

After obtaining this Global Relation we attempt to obtain the Ehrenpreis form for this expression. To do this we first solve for $Q(x,t)$. We obtain the following expressions for (3.19) and (3.20) respectively,

$$Q(x,t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-ikx+\Lambda(k)t}\hat{Q}_0(k)dk - \frac{1}{2\pi}\int_{-\infty}^\infty e^{-ikx+\Lambda(k)t}G(k,t)dk, \tag{3.27}$$

and

$$Q(x,t) = \frac{1}{2\pi}\int_{-\infty}^\infty P^{-1}(k)e^{-ikx+J(k)t}P(k)\hat{Q}_0(k)dk \\ - \frac{1}{2\pi}\int_{-\infty}^\infty P^{-1}(k)e^{-ikx+J(k)t}P(k)\tilde{G}(k,t)dk. \tag{3.28}$$

Using (3.28) we see that our formulae for $q(x,t)$ and $p(x,t)$ are the following,

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-ikx+k^2t}\hat{q}_0 dk - \frac{1}{2\pi}\int_{-\infty}^\infty e^{-ikx+k^2t}(ik(f_0(q,0,t,k) \\ + f_0(p,0,t,k)) + f_1(q,0,t,k) + f_1(p,0,t,k))dk, \tag{3.29}$$

$$p(x,t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-ikx+k^2t}(\hat{q}_0 + \hat{p}_0)dk - \frac{1}{2\pi}\int_{-\infty}^\infty e^{-ikx+k^2t}(ik(2f_0(q,0,t,k) \\ + f_0(p,0,t,k)) + 2f_1(q,0,t,k) + f_1(p,0,t,k))dk. \tag{3.30}$$

We define the D^+ to be,

$$D^+ = \bigcup_{j=1}^{N} \{k \in \mathbb{C} : \operatorname{Im} k > 0, \operatorname{Re} \omega(k) < 0\}. \tag{3.31}$$

For $k \in \mathbb{C} \setminus D^+$, the integrand of the second term decays exponentially as $k \to \infty$ and due to Jordan's Lemma and Cauchy's Theorem, we can rewrite (3.27) - (3.30) as

$$Q(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx+\Lambda(k)t} \hat{Q}_0(k) dk - \frac{1}{2\pi} \int_{\partial D^+} e^{-ikx+\Lambda(k)t} G(k,t) dk, \tag{3.32}$$

$$\begin{aligned} Q(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} & P^{-1}(k) e^{-ikx+J(k)t} P(k) \hat{Q}_0(k) dk \\ & - \frac{1}{2\pi} \int_{\partial D^+} P^{-1}(k) e^{-ikx+J(k)t} P(k) \tilde{G}(k,t) dk, \end{aligned} \tag{3.33}$$

$$\begin{aligned} q(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx+k^2 t} \hat{q}_0 dk - \frac{1}{2\pi} \int_{\partial D^+} & e^{-ikx+k^2 t} (ik(f_0(q,0,t,k) \\ & + f_0(p,0,t,k)) + f_1(q,0,t,k) + f_1(p,0,t,k)) dk. \end{aligned} \tag{3.34}$$

$$\begin{aligned} p(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx+k^2 t} (\hat{q}_0 + \hat{p}_0) dk - \frac{1}{2\pi} \int_{\partial D^+} & e^{-ikx+k^2 t} (ik(2f_0(q,0,t,k) \\ & + f_0(p,0,t,k)) + 2f_1(q,0,t,k) + f_1(p,0,t,k)) dk. \end{aligned} \tag{3.35}$$

There are no singularities or branch points, so the integration path need not be deformed for this example. However, other more complicated examples with more equations in the system might have branch points which require the deformation of the contour.

3.8 Boundary Conditions and Dealing with Unknowns

For this example we assume our boundary conditions to be

$$q(0,t) = bp(0,t), \quad \text{and} \quad q_x(0,t) = \beta p_x(0,t). \tag{3.36}$$

This means that all our equations can be expressed in terms of two unknowns only, $f_0(p,0,t,k)$ and $f_1(p,0,t,k)$. To deal with these unknowns we first apply a map to the Global Relation which leaves these unknowns invariant.

For these unknowns the maps, $k \to k$ and $k \to -k$ leave them invariant and thus by applying the second one, we get a Global Relations valid in the upper half of the plane.

The new global relations are

$$e^{k^2t}(\hat{q}(-k,t)+\hat{p}(-k,t))-(\hat{q}_0(-k)+\hat{p}_0(-k))+ik(b+1)f_0(p,0,t,k) \\ -(\beta+1)f_1(p,0,t,k)=0 \quad (3.37)$$

and

$$e^{k^2t}\hat{q}(-k,t)-\hat{q}_0(-k)+ikf_0(q,0,t,k)-f_1(q,0,t,k)=0. \quad (3.38)$$

We can then apply Cramer's Rule to this two equation system containing two unknowns to get the following expressions for our unknowns,

$$f_0(p,0,t,k)=\frac{e^{k^2t}(\beta\hat{p}(-k,t)-\hat{q}(-k,t))-(\beta\hat{p}_0(-k)-\hat{q}_0(-k))}{ik(b-\beta)}, \quad (3.39)$$

$$f_1(p,0,t,k)=\frac{e^{k^2t}(b\hat{p}(-k,t)-\hat{q}(-k,t))-(b\hat{p}_0(-k)-\hat{q}_0(-k))}{(b-\beta)}. \quad (3.40)$$

We can resubstitute this into the Ehrenpreis Form to get a final expression independent of boundary values and perform asymptotic analysis to show the decay of the second integral. Thus, we get

$$q(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ikx+k^2t}\hat{q}_0dk-\frac{1}{2\pi}\int_{\partial D^+}e^{-ikx+k^2t} \\ (ik(b+1)\left(\frac{e^{k^2t}(\beta\hat{p}(-k,t)-\hat{q}(-k,t))-(\beta\hat{p}_0(-k)-\hat{q}_0(-k))}{ik(b-\beta)}\right) \\ +(\beta+1)\left(\frac{e^{k^2t}(b\hat{p}(-k,t)-\hat{q}(-k,t))-(b\hat{p}_0(-k)-\hat{q}_0(-k))}{(b-\beta)}\right))dk, \quad (3.41)$$

$$p(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ikx+k^2t}(\hat{q}_0+\hat{p}_0)dk-\frac{1}{2\pi}\int_{\partial D^+}e^{-ikx+k^2t} \\ (ik(2b+1)\left(\frac{e^{k^2t}(\beta\hat{p}(-k,t)-\hat{q}(-k,t))-(\beta\hat{p}_0(-k)-\hat{q}_0(-k))}{ik(b-\beta)}\right) \\ +(2\beta+1)\left(\frac{e^{k^2t}(b\hat{p}(-k,t)-\hat{q}(-k,t))-(b\hat{p}_0(-k)-\hat{q}_0(-k))}{(b-\beta)}\right))dk. \quad (3.42)$$

3.9 Conclusion

After conducting asymptotic analysis, we would have completed this UTM. However, this only serves as the base case for systems. Further cases need to be evaluated, such as those having multiple dispersion branches for the dispersion relation in higher dimensional systems. However, whether those exist for non-diagonalizable systems are yet to be determined.

In case that does not exist, there is the possibility of using an inductive argument can be used to prove that there are only polynomial eigenvalues for non-diagonalizable matrices, using the proof displayed in another paper as the base case.