Interface Linearization of The Nonlinear Schrödinger Equation

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Abstract

The nonlinear Schrödinger equation is one of the most salient expressions governing, in general, systems exhibiting nonlinear, dispersive or energy-preserving properties. In this paper, we formulate a linearised model of the NLS equation. Our model linearises the NLS equation about a step function approximation of the "bright" soliton in such a way as to make the PDE constant coefficient whilst modelling the nonlinear properties of the NLS equation. Finally, we apply the Unified Transform Method to the interface linear model of the NLS equation to present fully explicit solutions for the problem. Deconinck and Sheils's[1] implementation of the Unified Transform Method is instrumental in solving the interface IVP.

1 Linear Model of The NLS Equation

The focusing NLS equation is given by

$$iu_t + u_{xx} + 2|u|^2 u = 0, (x,t) \in \mathbb{R} \times (0,\infty). \tag{1}$$

It has a special solution known as a 'bright' soliton which is given by

$$\eta sech[\eta(x + 2\xi t - x_0)]e^{-i\Theta}, \text{where } \Theta = \xi x + (\xi^2 - \eta^2)t + \Theta_0{}^2 \tag{2}$$

We aim to linearise the NLS equation about a step function approximation of the 'bright' soliton rendering the NLS equation an interface IVP with constant coefficients. To this end, we substitute the step function for the nonlinear term in the NLS equation, $|u|^2$.
Though we may use a step function with n jumps, or interfaces, we shall restrict our model to a box step function approximation with 2 jumps. We consider the following linear model of the NLS equation:

$$iu_t^{(j)} + u_{xx}^{(j)} + 2\alpha_j u^{(j)} = 0, (x,t) \in \mathbb{R} \times (0,\infty),\ 1 \le j \le 3 \tag{3}$$

[1] [DS2020a]
[2] [Abl2011a, page 153].

where

$$\alpha(x) = \begin{cases} \alpha_1, & x < x_1 \\ \alpha_2, & x_1 < x < x_2 \\ \alpha_3, & x > x_2 \end{cases}$$

$$j \equiv \begin{cases} 1, & x < x_1 \\ 2, & x_1 < x < x_2 \\ 3, & x > x_2 \end{cases}$$

$$\text{and } \mathrm{x}_1 < ct < x_2 \quad (4)$$

with initial condition $u(x,0) = u_0(x)$ with step-like initial datum that satisfies

$$lim_{x\to-\infty}\partial_x^{(k)}u(x,t) = 0, \quad lim_{x\to+\infty}\partial_x^{(k)}u(x,t) = 0,$$

$0 \leq k \leq 1$

Given our discontinuities in the coefficient of the u term, we coerce our solution and its first derivative to be continuous in x at the 2 interfaces, x_1 and x_2.

Defining a Linear Moving-Interface PDE Problem

Our linear model of the NLS equation is defined in the lab frame. To easily find explicit solutions to the time dependent PDE, we need to observe the equation in the body/ travelling frame so as to keep the interfaces fixed relative to the step function soliton approximation. Since a soliton maintains its shape, besides travelling at constant velocity, a simple translation is all that is required to relate the lab frame to the travelling frame. Here is the translation and its corollaries:

$$q(x,t) = u(x+ct,t) \text{ where c ¿ 0}$$
$$q_t = cu_x + u_t, \quad q_x = u_x$$

c is the soliton's front speed. We may leave it as a free parameter. In our case, we choose $c = 2\xi$. Given our transition to the moving frame, our PDE becomes

$$iq_t^{(j)} - icq_x^{(j)} + q_{xx}^{(j)} + 2\alpha_j^{'}u^{(j)} = 0, (x,t) \in \mathbb{R} \times (0,\infty), 1 \le j \le 3 \tag{5}$$

where

$$\alpha^{'}(x) = \begin{cases} \alpha_1^{'}, & x < x_1 - ct \\ \alpha'_2, & x_1 - ct < x < x_2 - ct \\ \alpha_3^{'}, & x > x_2 - ct \end{cases}$$

$$\text{and } \mathrm{x}_1 - ct < 0 < x_2 \tag{6}$$

To simplify notation, $x_k - ct \equiv x_k^{'}$ for $1 \le k \le 2$. Equation (5) satisfies the following conditions

$$lim_{x\to-\infty}\partial_x^{(k)}q(x,t) = 0, \quad lim_{x\to+\infty}\partial_x^{(k)}q(x,t) = 0,$$

$0 \le k \le 1$
(7)

$$q^{(j)}(x,0) = q_0^{(j)}(x) = u_0^{(j)}(x) \tag{8}$$

To keep everything explicit, we choose

$$u_0^{(j)}(x) = \begin{cases} b, & x_1^{'} < x < x_2^{'} \\ 0, & \text{elsewhere} \end{cases} \tag{9}$$

and define the following continuity interface conditions

$$\partial_x^{(k)}q^j(x_j^{'},t) = \partial_x^{(k)}q^{j+1}(x_j^{'},t) = 0,$$

$$0 \le k \le 1 \text{ , } 1 \le j \le 3, \ t > 0 \tag{10}$$

Note that since we are linearising about the step-function approximation of the 'bright' soliton, $u_0^{(j)}(x) = \alpha_j = \alpha_j^{'}$.

We shall define conditions for our PDE problem later. For now, let us re-express our PDE problem to simplify solving it with the Unified Transform Method. We observe that our PDE, equation(5), has 2 unique spatial differential operators. We may multiply our PDE by an appropriate factor to reduce our PDE to having only 1 unique spatial differential operators, i.e. the ∂_{xx} operator. The reason for this somplification will become apparent in the next section where we apply the UTM to our PDE problem. To this end, we use the following transformation and its corollaries:

$$Q(x,t) = e^{-icx/2}q(x,t), \quad Q_t = e^{-icx/2}q_t$$
$$Q_x = e^{-icx/2}(-icq/2 + q_x), \quad Q_{xx} = e^{-icx/2}(-c^2q/4 - icq_x + q_{xx})$$

After making the necessary substitutions to (5),especially the q_x and q_{xx}, our PDE problem becomes

$$iQ_t^{(j)} + Q_{xx}^{(j)} + (c^2/4 + 2\alpha_j^{'})Q^{(j)} = 0, (x,t) \in \mathbb{R} \times (0,\infty), 1 \leq j \leq 3 \tag{11}$$

where

$$\alpha^{'}(x) = \begin{cases} \alpha_1^{'} = 0, & x < x_1^{'} \\ \alpha'_2 = b, & x_1^{'} < x < x_2^{'} \\ \alpha_3^{'} = 0, & x > x_2^{'} \end{cases} \tag{12}$$

with the following conditions:

$$lim_{x\to-\infty}\partial_x^{(k)}Q(x,t) = 0, \quad lim_{x\to+\infty}\partial_x^{(k)}Q(x,t) = 0,$$
$$0 \leq k \leq 1 \tag{13}$$

$$Q^{(j)}(x,0) = Q_0^{(j)}(x) = u_0^{(j)}(x) \tag{14}$$

$$Q^{(1)}(x_1^{'},t) = Q^{(2)}(x_1^{'},t), \quad Q^{(2)}(x_2^{'},t) = Q^{(3)}(x_2^{'},t)$$
$$Q_x^{(j)}(x_j^{'},t) = Q_x^{(j+1)}(x_j^{'},t) + ic/2[Q^{(j+1)}(x_j^{'},t) - Q^{(j)}(x_j^{'},t)] \tag{15}$$

2 Solving Interface IVP using The Unified Transform Method

In this section, we apply the UTM to our PDE problem to arrive at a solution. It is important to note that we have 3 unique j domains arising from our two interfaces. 2 semi-infinite domains sandwich 1 finite one. We shall essentially apply the UTM to 3 PDEs, each respective to its domain and use the continuity interface conditions to derive explicit simultaneous solutions to all 3 PDEs.

2.1 Global Relation for Domain (1)

We consider the PDE (11) for domain $j \equiv 1$. Substituting 0 for $\alpha_1^{'}$ and multiplying both sides by $-i$, our PDE becomes

$$Q_t^{(1)}(x,t) - iQ_{xx}^{(1)}(x,t) - \frac{ic^2}{4}Q^{(1)}(x,t) = 0, (x,t) \in (-\infty, x_1^{'}) \times (0,\infty) \quad (16)$$

First, we need to apply the Fourier Transform, given by

$$\mathcal{F}(\phi)(\lambda) = \hat{\phi}(\lambda) = \int_{-\infty}^{\infty} \mathrm{e}^{-i\lambda t}\phi(x)\, dx \quad (17)$$

to (16), choosing the appropriate limits, $-\infty$ and $x_1^{'}$. Note that we can take the time derivative outside the transform. Ergo, our transformed PDE is

$$\frac{d}{dt}\hat{Q}_t^{(1)}(\lambda,t) - i\hat{Q}_{xx}^{(1)}(\lambda,t) - \frac{ic^2}{4}\hat{Q}^{(1)}(\lambda,t) = 0 \quad (18)$$

The Fourier Transform diagonalises ∂_{xx} operator into a multiplicative scalar. We can observe this by integrating $\hat{Q}_{xx}$ by parts:

$$\hat{Q}_{xx}^{(1)}(\lambda) = \int_{-\infty}^{x_1^{'}} \mathrm{e}^{-i\lambda t}Q_{xx}^{(1)}(x)\, dx = \mathrm{e}^{-i\lambda t}Q_x^{(1)}(x)\Big| x = -\infty^{x} = x_1^{'}$$
$$+ i\lambda \left[\mathrm{e}^{-i\lambda t}Q^{(1)}(x)\Big| x = -\infty^{x} = x_1^{'} + i\lambda \int_{-\infty}^{x_1^{'}} \mathrm{e}^{-i\lambda t}Q^{(1)}(x)\, dx \right] \quad (19)$$

Notice that the last term of (19) is equivalent to $\hat{Q}^{(1)}(\lambda)$. Applying the boundary conditions from (13) and factoring out $\hat{Q}^{(1)}$, (18)*becomes* $\left(\frac{d}{dt} - \frac{ic^2}{4} + i\lambda^2\right)\hat{Q}^{(1)}(\lambda,t) = i\mathrm{e}^{-i\lambda x_1^{'}}\left[Q_x^{(1)}(x_1^{'},t) + i\lambda Q^{(1)}(x_1^{'},t)\right]$ (20)

By reversing chain rule for differentiation, we may factor out $\frac{d}{dt}$ from the LHS of (20) and multiply both sides by $\mathrm{e}^{i(\lambda^2-\frac{c^2}{4})t}$. We thus have

$$\frac{d}{dt}\left(\mathrm{e}^{i(\lambda^2-\frac{c^2}{4})t}\hat{Q}^{(1)}(\lambda,t)\right) = i\mathrm{e}^{-i\lambda x_1^{'}}\left[\mathrm{e}^{i(\lambda^2-\frac{c^2}{4})t}Q_x^{(1)}(x_1^{'},t) + i\lambda\mathrm{e}^{i(\lambda^2-\frac{c^2}{4})t}Q^{(1)}(x_1^{'},t)\right] \tag{21}$$

Next, we integrate (21) w.r.t. time to get

$$\mathrm{e}^{i(\lambda^2-\frac{c^2}{4})t}\hat{Q}^{(1)}(\lambda,t) - \hat{Q}^{(1)}(\lambda,0) = i\mathrm{e}^{-i\lambda x_1^{'}}\left[f_1^{(1)}(\lambda,x_1^{'},t) + i\lambda f_0^{(1)}(\lambda,x_1^{'},t)\right]$$
$$\text{where } f_k^{(1)}(\lambda,x,t) = \int_0^t \mathrm{e}^{i(\lambda^2-\frac{c^2}{4})s}\partial_x^{(k)}Q^{(1)}(x,s)\,ds \tag{22}$$

We apply (14) and (9) to the second term on the LHS of (22). We also note that $\hat{Q}^{(1)}(\lambda,t)$ only converges $\forall\ \lambda \in^{+}$ because $x_1^{'} < 0$. Finally, we have the following global relation equation:

$$\mathrm{e}^{i(\lambda^2-\frac{c^2}{4})t}\hat{Q}^{(1)}(\lambda,t) = i\mathrm{e}^{-i\lambda x_1^{'}}\left[f_1^{(1)}(\lambda,x_1^{'},t) + i\lambda f_0^{(1)}(\lambda,x_1^{'},t)\right],\ \forall\ \lambda \in \mathbb{C}^{+} \tag{23}$$

2.2 Global Relation for Domain (2)

We consider the PDE (11) for domain $j \equiv 2$. Substituting b for $\alpha_2^{'}$ and multiplying both sides by $-i$, our PDE becomes

$$Q_t^{(2)}(x,t) - iQ_{xx}^{(2)}(x,t) - i\left(\frac{c^2}{4} + 2b\right)Q^{(2)}(x,t) = 0, (x,t) \in (x_1^{'},x_2^{'}) \times (0,\infty) \tag{24}$$

We may obtain the global relation for domain 2 using similar arguments as those in domain 1, being mindful of index notation and conditions provided for our PDE (11). It is worth noting that this domain is a finite one with no asymptotic conditions. At this juncture, we only use the initial condition for PDE (11):

$$\hat{Q}^{(2)}(\lambda, 0) = \hat{u_0}^{(2)}(\lambda) = -bi\lambda\left[\mathrm{e}^{-i\lambda x_2'} - \mathrm{e}^{-i\lambda x_1'}\right] \tag{25}$$

We have the following definition for our unknown integrals:

$$f_k^{(2)}(\lambda, x, t) = \int_0^t \mathrm{e}^{i(\lambda^2 - \frac{c^2}{4} - 2b)s} \partial_x^{(k)} Q^{(2)}(x, s)\, ds \tag{26}$$

Considering all this, we obtain the following global relation equation:

$$\begin{aligned}\mathrm{e}^{i(\lambda^2 - \frac{c^2}{4} - 2b)t} \hat{Q}^{(2)}(\lambda, t) = \hat{u_0}^{(2)}(\lambda) + i\mathrm{e}^{-i\lambda x_2'}\left[f_1^{(2)}(\lambda, x_2', t) + i\lambda f_0^{(2)}(\lambda, x_2', t)\right] \\ - i\mathrm{e}^{-i\lambda x_1'}\left[f_1^{(2)}(\lambda, x_1', t) + i\lambda f_0^{(2)}(\lambda, x_1', t)\right], \ \forall\ \lambda \in \mathbb{C} \end{aligned} \tag{27}$$

2.3 Global Relation for Domain (3)

We consider the PDE (11) for domain $j \equiv 3$. Substituting 0 for α_3' and multiplying both sides by $-i$, our PDE becomes

$$Q_t^{(3)}(x, t) - iQ_{xx}^{(3)}(x, t) - \frac{ic^2}{4} Q^{(3)}(x, t) = 0, (x, t) \in (x_2', \infty) \times (0, \infty) \tag{28}$$

Due to the reflection symmetry between the PDE problems for domain 1 and 3 given our linearisation about a box step function, we may obtain the global relation for domain 3 using similar arguments as those in domain 1. We simply need to change the indexing to $j \equiv 3$ and choose the appropriate limits for x, i.e. $x_2' < x < \infty$ where necessary. Otherwise, our definitions for domain 3 are essentially the same as those in domain 1. Therefore, our global relation equation for domain 3 is given by

$$\mathrm{e}^{i(\lambda^2 - \frac{c^2}{4})t} \hat{Q}^{(3)}(\lambda, t) = -i\mathrm{e}^{-i\lambda x_2'}\left[f_1^{(3)}(\lambda, x_2', t) + i\lambda f_0^{(3)}(\lambda, x_2', t)\right], \ \forall\ \lambda \in \mathbb{C}^- \tag{29}$$

2.4 Dispersion Relations

In this section, we consider our GR equations obtained in the previous sections. We redefine our unknown integrals, $f_k^{(j)}$s, to depend on the dispersion relations of their respective domains, $\omega_j(\lambda)$. We then need to redefine our GR equations to have the $f_k^{(j)}$s depend on the dispersion relations of their respective domains. We aim to transform our GR equations such that the $f_k^{(j)}$s depend on a common argument. The choice of common argument is arbitrary but some choices allow for simpler calculations. We will be left with a linear system of transformed GR equations.

Observing the definitions for our $f_k^{(j)}$s, we can read off the dispersion relations $\omega_j(\lambda)$s from the power of e. We thus have the following $\omega_j(\lambda)$s:

$$\omega_1(\lambda) = i(\lambda^2 - \tfrac{c^2}{4}), \quad \omega_2(\lambda) = i(\lambda^2 - \tfrac{c^2}{4} - 2b), \quad \omega_3(\lambda) = i(\lambda^2 - \tfrac{c^2}{4}) \tag{30}$$

With this information, we may redefine our $f_k^{(j)}$s to encode our PDE's continuity interface conditions (15)

$$f_0^{(j)}(\omega, t) = \int_0^t \mathrm{e}^{\omega s} Q^{(j)}(x_j', s)\, ds = \int_0^t \mathrm{e}^{\omega s} Q^{(j+1)}(x_j', s)\, ds \tag{31}$$

$$f_1^{(j)}(\omega, t) = \int_0^t \mathrm{e}^{\omega s} Q_x^{(j)}(x_j', s)\, ds = \int_0^t \mathrm{e}^{\omega s} Q_x^{(j+1)}(x_j', s) + \tfrac{ic}{2}\left[Q^{(j+1)}(x_j', s) - Q^{(j)}(x_j', s)\right] ds \tag{32}$$

for $t > 0, \ \omega \in \mathbb{C}$.

Next, we redefine our GR equations to show dependence on $\omega_j(\lambda)$s.

$$\mathrm{e}^{\omega_1 t}\hat{Q}^{(1)}(\lambda, t) = i\mathrm{e}^{-i\lambda x_1'}\left[f_1^{(1)}(\omega_1, t) + i\lambda f_0^{(1)}(\omega, t)\right], \ \forall\ \lambda \in \mathbb{C}^+ \tag{33}$$

$$\begin{aligned}\mathrm{e}^{\omega_2 t}\hat{Q}^{(2)}(\lambda, t) = \hat{u_0}^{(2)}(\lambda) &+ i\mathrm{e}^{-i\lambda x_2'}\left[f_1^{(1)}(\omega_2, t) + (i\lambda - \tfrac{ic}{2} + 1) f_0^{(1)}(\omega_2, t)\right] \\ &- i\mathrm{e}^{-i\lambda x_1'}\left[f_1^{(1)}(\omega_2, t) + (i\lambda - \tfrac{ic}{2} + 1) f_0^{(1)}(\omega_2, t)\right], \ \forall\ \lambda \in \mathbb{C}\end{aligned} \tag{34}$$

$$\mathrm{e}^{\omega_3 t}\hat{Q}^{(3)}(\lambda, t) = -i\mathrm{e}^{-i\lambda x_2'}\left[f_1^{(2)}(\omega_3, t) + (i\lambda - \tfrac{ic}{2} + 1) f_0^{(2)}(\omega_3, t)\right], \ \forall\ \lambda \in \mathbb{C}^- \tag{35}$$

where $\hat{u_0}^{(2)}(\lambda)$ is given by equation (25).

2.5 Global Relation Transformations

For simplicity, we choose the common argument for our $f_k^{(j)}(\omega_j)$s to be $\omega_{1,3}(\lambda) = i(\lambda^2 - \frac{c^2}{4})$. Therefore, $\omega_j \mapsto i(\lambda^2 - \frac{c^2}{4}) \equiv \gamma$ To transform our GR equations, we choose the following map for λ:

$$\lambda \mapsto \pm\nu^{(j)}(\lambda) \equiv \lambda\sqrt{1 + \frac{2\alpha_j}{\lambda^2}} \tag{36}$$

We observe that each GR equation will be transformed twice. Therefore, our transformed GR equations are

$$\mathrm{e}^{\gamma t}\hat{Q}^{(1)}(\nu^{(1)}, t) = i\mathrm{e}^{-i\nu^{(1)}x_1'}\left[f_1^{(1)}(\gamma, t) + i\nu^{(1)}f_0^{(1)}(\gamma, t)\right] \tag{37}$$

$$\mathrm{e}^{\gamma t}\hat{Q}^{(1)}(-\nu^{(1)}, t) = i\mathrm{e}^{i\nu^{(1)}x_1'}\left[f_1^{(1)}(\gamma, t) - i\nu^{(1)}f_0^{(1)}(\gamma, t)\right] \tag{38}$$

$$\begin{aligned}\mathrm{e}^{\gamma t}\hat{Q}^{(2)}(\nu^{(2)}, t) = \hat{u_0}^{(2)}(\nu^{(2)}) + i\mathrm{e}^{-i\nu^{(2)}x_2'}&\left[f_1^{(1)}(\gamma, t) + (i\nu^{(2)} - \tfrac{ic}{2} + 1)f_0^{(1)}(\gamma, t)\right]\\ - i\mathrm{e}^{-i\nu^{(2)}x_1'}&\left[f_1^{(1)}(\gamma, t) + (i\nu^{(2)} - \tfrac{ic}{2} + 1)f_0^{(1)}(\gamma, t)\right]\end{aligned} \tag{39}$$

$$\begin{aligned}\mathrm{e}^{\gamma t}\hat{Q}^{(2)}(-\nu^{(2)}, t) = \hat{u_0}^{(2)}(-\nu^{(2)}) + i\mathrm{e}^{i\nu^{(2)}x_2'}&\left[f_1^{(1)}(\gamma, t) + (-i\nu^{(2)} - \tfrac{ic}{2} + 1)f_0^{(1)}(\gamma, t)\right]\\ - i\mathrm{e}^{i\nu^{(2)}x_1'}&\left[f_1^{(1)}(\gamma, t) + (-i\nu^{(2)} - \tfrac{ic}{2} + 1)f_0^{(1)}(\gamma, t)\right]\end{aligned} \tag{40}$$

$$\mathrm{e}^{\gamma t}\hat{Q}^{(3)}(\nu^{(3)}, t) = -i\mathrm{e}^{-i\nu^{(3)}x_2'}\left[f_1^{(2)}(\gamma, t) + (i\nu^{(3)} - \tfrac{ic}{2} + 1)f_0^{(2)}(\gamma, t)\right] \tag{41}$$

$$\mathrm{e}^{\gamma t}\hat{Q}^{(3)}(-\nu^{(3)}, t) = -i\mathrm{e}^{i\nu^{(3)}x_2'}\left[f_1^{(2)}(\gamma, t) + (-i\nu^{(3)} - \tfrac{ic}{2} + 1)f_0^{(2)}(\gamma, t)\right] \tag{42}$$

2.6 Solving for unknowns using Cramer's Rule

We consider our transformed GR equations. We first need to find the regions of validity for each equation. We then use Cramer's rule to simultaneously solve for unknowns in any pair of GR equations with overlapping regions of validity. To check the regions of validity for each GR equation, we check for convergence of $\hat{Q}^{(j)}(\pm\nu^{(j)})$. We thus find that (37) and (42) are valid in $\mathbb{C}^+$, (38) and (41) are valid in $\mathbb{C}^-$ and (39) and (40) are valid in $\mathbb{C}$. Considering these regions of validity, we set up linear system of equations to solve for the unknowns $f_k^{(l)}, 0 \leq k \leq 1, 1 \leq l \leq 2$.

It appears that (39) and (40) are incomplete or incorrect because they lack $f_k^{(2)}, 1 \leq k \leq 2$. *We are therefore unable to set up a linear system of equations to solve for unknowns.*
Correction pending...

2.7 Ehrenpreis Equation for Domain (1)

We consider our global relation equation (33) and apply the inverse Fourier Transform defined as

$$\mathcal{F}^{-}\mathcal{F}(\phi)(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{i\lambda t}\phi(\lambda)\, d\lambda \tag{43}$$

We then solve for $Q^{(1)}(x,t)$ to get

$$2\pi Q^{(1)}(x,t) = i\int_{-\infty}^{+\infty} \mathrm{e}^{i\lambda(x-x_1')-\omega_1 t}[f_1^{(1)}(\omega_1,t) + i\lambda f_0^{(1)}(\omega_1,t)]\, d\lambda \tag{44}$$

We then aim to deform the contour of integration for the unknowm integrand away from $\mathbb{R}$. To this end, we make the following definitions:

$$D \coloneqq \{\lambda \in \mathbb{C} : \pm\mathrm{Re}(\omega_1) < 0\}, \quad D^{\pm} \coloneqq D \cap \mathbb{C}^{\pm} \tag{45}$$

$$E \coloneqq \{\lambda \in \mathbb{C} : \pm\mathrm{Re}(\omega_1) > 0\}, \quad E^{\pm} \coloneqq D \cap \mathbb{C}^{\pm} \tag{46}$$

To find the transition lines between D and E regions, we solve for θ in $\mathrm{Re}(\omega_1) = 0$. Reexpressing ω_1 in polar form makes θ more apparent. In this case, $\theta = \frac{\pi}{2}$. Use a probe point to test, using the definitions above, which region is which in the complex space.

Deforming Contours using Jordan's Lemma and Cauchy's Theorem

Jordan's lemma is given by

$$\lim_{R\to\infty}\int_{\partial R} \mathrm{e}^{-i\lambda z} f(z)dz = 0 \tag{47}$$

where ∂R is a circular arc and $f(z) \to 0$ as R $\to \infty$. Jordan's lemma is defined in $\mathbb{C}^{\pm}$ depending on whether z is positive or negative respectively. Cauchy's theorem allows us to deform a contour of integration given that the integrand is holomorphic entirely in the region enclosed by the contour. We apply both to deform the contours. To this end, we use integration by parts to check for decay of

$$e^{-\omega_1 t} f_k^{(1)}(\omega_1, t),\ 0 \leq k \leq 1. \tag{48}$$

To wit,

$$\int_0^t \mathrm{e}^{-i\omega_1(t-s)}\partial_x^{(k)}Q^{(1)}(x_1', s)\, ds = \omega_1^{-1}\left(\mathrm{e}^{-i\omega_1(t-s)}\partial_x^{(k)}Q^{(1)}(x_1', s)\Big|_{s=0}^{s=t}\right) - \omega_1^{-1}\int_0^t \mathrm{e}^{-i\omega_1(t-s)}\partial_t\partial_x^{(k)}Q^{(1)}(x_1', s)\, ds \tag{49}$$

We thus have the first term being $\mathcal{O}(|\omega_1|^{-1})$ uniformly in $arg(\lambda)$ as $\lambda \to \infty$ within clos(E). The integrand of the second term has $\mathcal{O}(1)$ uniformly in $arg(\lambda)$ by Riemann-Lebesgue lemma. Note that E is chosen so that $\mathrm{e}^{-i\omega_1(t-s)} = \mathcal{O}(1)$ for $s \in [0, t]$ because $\mathrm{Re}(\omega_1) > 0$ in E. Therefore,

$$\mathrm{e}^{-i\omega_1 t}[f_1^{(1)}(\omega_1, t) + i\lambda f_0^{(1)}(\omega_1, t)] = \mathcal{O}(|\omega|^{-1})\text{uniformly in } arg(\lambda) \text{ as } \lambda \to \infty \text{ within } [E] \tag{50}$$

Therefore, by Jordan's lemma,

$$\int_{\partial E^-} e^{i\lambda(x-x_1')-\omega_1 t}[f_1^{(1)}(\omega_1, t) + i\lambda f_0^{(1)}(\omega_1, t)]\, d\lambda = 0,\ \forall (x - x_1') < 0 \tag{51}$$

Note that the contour is along ∂E^- because Jordan's lemma is defined on $\mathbb{C}^-$. This is because $(x - x_1') < 0$. Recall that $x_1' < 0$. To move our contour away from $\mathbb{R}$, we note that

$$\int_{-\infty}^{\infty} \dots \, d\lambda = \left[\int_{-\infty}^{\infty} - \int_{\partial E^-} \right] \dots \, d\lambda = \int_{\partial D^-} \dots \, d\lambda \tag{52}$$

We can now write our $EFt^{(1)}$ equation:

$$2\pi Q^{(1)}(x,t) = i \int_{\partial D^-} e^{i\lambda(x-x_1^{'})-\omega_1 t} \left[f_1^{(1)}(\omega_1,t) + i\lambda f_0^{(1)}(\omega_1,t) \right] \, d\lambda \tag{53}$$

To get the $EF\tau^{(1)}$, we use a similar argument as that of deforming the contour of integration in (44). A trick we may use is to substitute $\int_t^\tau$ for $\int_0^t$ as the limits of integration of $f_k^{(j)}$. We show that

$$\mathrm{e}^{-i\omega_1 t} \left[\int_t^\tau \mathrm{e}^{\omega_1 s} Q^{(1)}(x_1^{'},s) \, ds + i\lambda \int_t^\tau \mathrm{e}^{\omega_1 s} \partial_x Q^{(1)}(x_1^{'},s) \, ds \right] = \mathcal{O}(|\omega|^{-1}) \tag{54}$$

uniformly in $arg(\lambda)$ as $\lambda \to \infty$ within $[D]$. Therefore, by Jordan's lemma,

$$\int_{\partial D^-} e^{i\lambda(x-x_1^{'})-\omega_1 t} \left[f_1^{(1)}(\omega_1,\tau) - f_1^{(1)}(\omega_1,t) + i\lambda (f_0^{(1)}(\omega_1,\tau) - f_0^{(1)}(\omega_1,t)) \right] \, d\lambda = 0 \tag{55}$$

We can now write our $EF\tau^{(1)}$ equation:

$$2\pi Q^{(1)}(x,t) = i \int_{\partial D^-} e^{i\lambda(x-x_1^{'})-\omega_1 t} \left[f_1^{(1)}(\omega_1,\tau) + i\lambda f_0^{(1)}(\omega_1,\tau) \right] \, d\lambda, \quad (x,t) \in (-\infty, x_1^{'}) \times [0,\tau] \tag{56}$$

2.8 Ehrenpreis Equation for Domain (2)

Since we suspect $GR^{(2)}$ to be incorrect, we hold off on writing its Ehrenpreis Form equation or making any corollary calculations partaining domain (2). Update pending...

2.9 Ehrenpreis Equation for Domain (3)

We consider our global relation equation (35). Again, the reflection symmetry between the PDE problems for domain 1 and 3 simplifies our work. We may obtain the Ehrenpreis equation for domain 3 using similar arguments as those in domain 1. We simply need to change the indexing to $j \equiv 3$ and choose the appropriate limits for x, i.e. $x_2^{'} < x < \infty$ where necessary. Otherwise, our definitions for domain 3 are essentially the same as those in domain 1. Therefore, our Ehrenpreis equation for domain 3, $EF^{(3)}$ is given by

$$2\pi Q^{(3)}(x,t) = -i \int_{\partial D^+} e^{i\lambda(x-x_2^{'})-\omega_3 t} \left[f_1^{(2)}(\omega_3,\tau) + (i\lambda - \tfrac{ic}{2} + 1) f_0^{(2)}(\omega_3,\tau) \right] d\lambda, \quad (x,t) \in (x_2^{'},\infty)\times[0,\tau] \tag{57}$$

2.10 Transforming Contours of Integration in EFτ Equations

Now that we have our $EF\tau$ equations, we need to once more use our λ transformation to have our unknowns depend on our common argument γ. This will require that we deform all integrals involving unknowns such that they integrate around a common region. To this end, we choose to deform our unknown integrals such that they integrate around region E^-[3]. Futhermore, we choose $\lambda = \nu^{(j)}(\lambda)$ on region D^+ and $\lambda = -\nu^{(j)}(\lambda)$ on region D^-. We thus have

$$2\pi Q^{(1)}(x,t) = i \int_{\partial E^-} e^{-i\nu^{(1)}(\lambda)(x-x_1^{'})+\gamma t} \left[\tfrac{\lambda}{\nu^{(1)}(\lambda)} f_1^{(1)}(\gamma,\tau) - i\lambda f_0^{(1)}(\gamma,\tau) \right] d\lambda, \quad (x,t) \in (-\infty,x_1^{'})\times[0,\tau] \tag{58}$$

$$2\pi Q^{(3)}(x,t) = -i \int_{\partial E^-} e^{i\nu^{(3)}(\lambda)(x-x_2^{'})+\gamma t} \left[\tfrac{\lambda}{\nu^{(3)}(\lambda)} f_1^{(2)}(\gamma,\tau) + (i\lambda - \tfrac{ic}{2} + 1) f_0^{(2)}(\gamma,\tau) \right] d\lambda, \quad (x,t) \in (x_2^{'},\infty)\times[0,\tau] \tag{59}$$

[3] [DS2020a]

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Disclaimer: This is an incomplete and slightly fragmented report. The solution to our PDE is at hand. However, correction is required on the PDE calculations for the finite domain, $j = 2$*. Update patch coming soon...*