Contents

Problem 1. Linearised KdV

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1 Problem set up

Consider the following Linearised KdV initial non-local value problem:

$$\begin{aligned}
q_t + q_{xxx} =& 0 & (x,t) \in& (0,L) \times (0,T), & (PDE)\\
q(1,t) =& 0 & t \in& [0,T], & (BC1)\\
q_x(1,t) =& 0 & t \in& [0,T], & (BC2)\\
\int_0^1 K(y)q(y,t)dy =& g_o(t) & t \in& [0,T], & (NLC)\\
q(x,0) =& q_0(x) & t \in& [0,1], & (IC)
\end{aligned}$$

where q_0, g_0 are known smooth functions.

We use $\widehat{\phi}$ to represent the Fourier transform of $\phi \in C^\infty[0,1]$:

$$\widehat{\phi}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x}\phi(x)dx$$

Use the following notation simplifications:

$$\begin{aligned}
f_j(\lambda; x, \tau) &= \int_0^\tau e^{-i\lambda^3 s}\partial_X^j q(x,s)ds\\
\widehat{q_0}(\lambda; y, z) &= \int_y^z e^{-i\lambda\xi} q_0(\xi)d\xi\\
\widehat{q}(\lambda, \tau; y, z) &= \int_y^z e^{-i\lambda\xi} q(\xi,\tau)d\xi.
\end{aligned}$$

2 Stage 1

Assume there exists $q : [0,1] \times [0,T] \to \mathbb{C}$, as smooth as we need, satisfying (PDE) and (IC). Apply Fourier transform to (PDE):

$$\begin{aligned}0 =& \widehat{q_t + q_{xxx}}(\lambda; t)\\ =& \widehat{q_t}(\lambda; t) + \widehat{q_{xxx}}(\lambda; t)\\ =& \partial_t \widehat{q}(\lambda; t) - i\lambda^3 \widehat{q}(\lambda; t) + e^{-i\lambda}\Big[\partial_{xx} q(1,t) + i\lambda \partial_x q(1,t) - \lambda^2 q(1,t)\Big] - \Big[\partial_{xx} q(0,t) + i\lambda \partial_x q(0,t) - \lambda^2 q(0,t)\Big].\end{aligned}$$

Rearrange and multiply by $e^{-i\lambda^3 t}$

$$\partial_t[e^{-i\lambda^3 t}\widehat{q}(\lambda; t)] = e^{-i\lambda^3 t}\Big[\partial_{xx} q(0,t) + i\lambda \partial_x q(0,t) - \lambda^2 q(0,t)\Big] - e^{-i\lambda - i\lambda^3 t}\Big[\partial_{xx} q(1,t) + i\lambda \partial_x q(1,t) - \lambda^2 q(1,t)\Big]$$

Integrate in time to solve the ODE for $\widehat{q}(\lambda; \cdot)$

$$\begin{aligned}e^{-i\lambda^3 t}\widehat{q}(\lambda; t) - \widehat{q}(\lambda; 0) = & \int_0^t e^{-i\lambda^3 s}\Big[\partial_{xx} q(0,s) + i\lambda \partial_x q(0,s) - \lambda^2 q(0,s)\Big] ds\\ & - e^{-i\lambda}\int_0^t e^{-i\lambda^3 s}\Big[\partial_{xx} q(1,s) + i\lambda \partial_x q(1,s) - \lambda^2 q(1,s)\Big] ds\end{aligned}$$

Note that (IC) implies $\widehat{q}(\lambda; 0) = \widehat{q_0}(\lambda)$. Also introduce notation

$$f_j(\lambda; X, t) := \int_0^t e^{-i\lambda^3 s}\partial_x^j q(X, s) ds$$

Then,

$$e^{-i\lambda^3 t}\widehat{q}(\lambda; t) - \widehat{q_0}(\lambda) = f_2(\lambda; 0, t) + i\lambda f_1(\lambda; 0, t) - \lambda^2 f_0(\lambda; 0, t) - e^{-i\lambda}\big[f_2(\lambda; 1, t) + i\lambda f_1(\lambda; 1, t) - \lambda^2 f_0(\lambda; 1, t)\big].$$

This is Global Relation (GR) valid $\forall t \in [0,T], \forall \lambda \in \mathbb{C}$. Now solve for $q(x,t)$. Rearranging

$$\begin{aligned}\widehat{q}(\lambda; t) =& e^{i\lambda^3 t}\widehat{q_0}(\lambda) + e^{i\lambda^3 t}\big[f_2(\lambda; 0, t) + i\lambda f_1(\lambda; 0, t) - \lambda^2 f_0(\lambda; 0, t)\big]\\ & - e^{-i\lambda + i\lambda^3 t}\big[f_2(\lambda; 1, t) + i\lambda f_1(\lambda; 1, t) - \lambda^2 f_0(\lambda; 1, t)\big]\end{aligned}$$

Applying inverse Fourier transform

$$\begin{aligned}2\pi q(x,t) = & \int_{-\infty}^{\infty} e^{i\lambda x + i\lambda^3 t}\widehat{q_0}(\lambda) d\lambda + \int_{-\infty}^{\infty} e^{i\lambda x + i\lambda^3 t}\big[f_2(\lambda; 0, t) + i\lambda f_1(\lambda; 0, t) - \lambda^2 f_0(\lambda; 0, t)\big] d\lambda\\ & - \int_{-\infty}^{\infty} e^{i\lambda(x-1) + i\lambda^3 t}\big[f_2(\lambda; 1, t) + i\lambda f_1(\lambda; 1, t) - \lambda^2 f_0(\lambda; 1, t)\big] d\lambda\end{aligned}$$

valid $\forall x \in (0,1), \forall t \in [0,T]$.

We aim to deform the latter two contours of integration away from $\mathbb{R}$. We need the following definitions:

$$\begin{aligned} \mathbb{C}^{\pm} &:= \{\lambda \in \mathbb{C} : \pm \mathrm{Im}(\lambda) > 0\} \\ D &:= \{\lambda \in \mathbb{C} : \mathrm{Re}(-i\lambda^3) < 0\} \\ D^{\pm} &:= D \cap \mathbb{C}^{\pm} \\ E &:= \{\lambda \in \mathbb{C} : \mathrm{Re}(-i\lambda^3) > 0\} \\ E^{\pm} &:= E \cap \mathbb{C}^{\pm} \end{aligned}$$

Orient the boundaries of these (unions of) sectors positively; the sector lies to the left of its boundary. Consider $\lambda \in \mathrm{clos(E)}$. We want to show that

$$\int_{\partial E^+} e^{i\lambda x + i\lambda^3 t}\big[f_2(\lambda;0,t) + i\lambda f_1(\lambda;0,t) - \lambda^2 f_0(\lambda;0,t)\big] d\lambda = 0$$

First we want to show that

$$\lim_{\lambda\to\infty} e^{i\lambda^3 t}\big[f_2(\lambda;0,t) + i\lambda f_1(\lambda;0,t) - \lambda^2 f_0(\lambda;0,t)\big] = 0$$

Consider

$$\begin{aligned} \lim_{\lambda\to\infty} e^{i\lambda^3 t} f_j(\lambda;X,t) &= \int_0^t e^{-i\lambda^3(s-t)}\partial_x^j q(X,s)ds \\ &= (-i\lambda)^{-3}\Big[\partial_x^j q(X,t) \underbrace{-e^{i\lambda^3 t}\partial_x^j q(X,0)}_{\text{decays as } \lambda\to\infty} - \int_0^t e^{-i\lambda^3(s-t)}\partial_t\partial_x^j q(X,s)ds\Big] \end{aligned}$$

Consider

$$\begin{aligned} \Big|\int_0^t e^{-i\lambda^3(s-t)}\partial_t\partial_x^j q(X,s)ds\Big| &\le \int_0^t \Big|e^{-i\lambda^3(s-t)}\Big|\Big|\partial_t\partial_x^j q(X,s)\Big|ds \\ &\le (t-0)\max_{s\in[0,t]}\big|e^{-i\lambda^3(s-t)}\big| \max_{s\in[0,t]}\big|\partial_t\partial_x^j q(X,s)\big| \\ &\le t\max_{s\in[0,t]}\big|\partial_t\partial_x^j q(X,s)\big| \qquad \forall\lambda\in clos(E) \end{aligned}$$

So, $e^{i\lambda^3 t} f_j(\lambda;X,t) = \mathcal{O}(|\lambda^{-3}|)$, uniformly in $\arg(\lambda)$ as $\lambda\to\infty$ within clos(E). Similarly, $i\lambda e^{i\lambda^3 t} f_j(\lambda;X,t) = \mathcal{O}(|\lambda^{-2}|)$ and $\lambda^2 e^{i\lambda^3 t} f_j(\lambda;X,t) = \mathcal{O}(|\lambda^{-1}|)$.

Then, $e^{i\lambda^3 t}\big[f_2(\lambda;X,t)+i\lambda f_1(\lambda;X,t)-\lambda^2 f_0(\lambda;X,t)\big] = \mathcal{O}(|\lambda^{-1}|)$, uniformly in $\arg(\lambda)$ as $\lambda\to\infty$ within clos(E). So, we can use Jordan's Lemma. Also, note that $e^{i\lambda^3 t}\big[f_2(\lambda;X,t) + i\lambda f_1(\lambda;X,t) - \lambda^2 f_0(\lambda;X,t)\big]$ is entire by Morera's theorem. So, we can use Cauchy's theorem. Hence, by Jordan's lemma and Cauchy's theorem,

$$\int_{\partial E^+} e^{i\lambda x + i\lambda^3 t}\big[f_2(\lambda;0,t) + i\lambda f_1(\lambda;0,t) - \lambda^2 f_0(\lambda;0,t)\big] d\lambda = 0$$

So,

$$\begin{aligned}&\int_{-\infty}^{\infty} e^{i\lambda x+i\lambda^3 t}\big[f_2(\lambda;0,t)+i\lambda f_1(\lambda;0,t)-\lambda^2 f_0(\lambda;0,t)\big]d\lambda\\ =&\left\{\int_{-\infty}^{\infty}-\int_{\partial E^+}\right\}e^{i\lambda x+i\lambda^3 t}\big[f_2(\lambda;0,t)+i\lambda f_1(\lambda;0,t)-\lambda^2 f_0(\lambda;0,t)\big]d\lambda\\ =&\int_{\partial D^+} e^{i\lambda x+i\lambda^3 t}\big[f_2(\lambda;0,t)+i\lambda f_1(\lambda;0,t)-\lambda^2 f_0(\lambda;0,t)\big]d\lambda.\end{aligned}$$

Similarly,

$$\begin{aligned}&-\int_{-\infty}^{\infty} e^{i\lambda(x-1)+i\lambda^3 t}\big[f_2(\lambda;1,t)+i\lambda f_1(\lambda;1,t)-\lambda^2 f_0(\lambda;1,t)\big]d\lambda\\ =&\left\{\int_{\infty}^{-\infty}-\int_{\partial E^-}\right\}e^{i\lambda(x-1)+i\lambda^3 t}\big[f_2(\lambda;1,t)+i\lambda f_1(\lambda;1,t)-\lambda^2 f_0(\lambda;1,t)\big]d\lambda\\ =&\int_{\partial D^-} e^{i\lambda(x-1)+i\lambda^3 t}\big[f_2(\lambda;1,t)+i\lambda f_1(\lambda;1,t)-\lambda^2 f_0(\lambda;1,t)\big]d\lambda.\end{aligned}$$

We have arrived at the Ehrenpreis form:

$$\begin{aligned}2\pi q(x,t)=&\int_{-\infty}^{\infty} e^{i\lambda x+i\lambda^3 t}\widehat{q_0}(\lambda)d\lambda+\int_{\partial D^+} e^{i\lambda x+i\lambda^3 t}\big[f_2(\lambda;0,t)+i\lambda f_1(\lambda;0,t)-\lambda^2 f_0(\lambda;0,t)\big]d\lambda\\ &+\int_{\partial D^-} e^{i\lambda(x-1)+i\lambda^3 t}\big[f_2(\lambda;1,t)+i\lambda f_1(\lambda;1,t)-\lambda^2 f_0(\lambda;1,t)\big]d\lambda\end{aligned}\tag{1}$$

valid $\forall x\in(0,1),\forall t\in[0,T]$.

3 Stage 2

Assume there exists q that satisfies not only (PDE) and (IC) but also (BC). Remember that the generalized global relation(GGR) is the same for non-local initial value problems as for the boundary value problems that have the same PDE. Hence, we already know the GGR:

$$\begin{aligned}\widehat{q_0}(\lambda;y,z)-e^{-i\lambda^3 t}\widehat{q}(\lambda,\tau;y,z)=&e^{-i\lambda y}\left(-\lambda^2 f_0(\lambda;y,\tau)+i\lambda f_1(\lambda;y,\tau)+f_2(\lambda;y,\tau)\right)\\ &-e^{-i\lambda z}\left(-\lambda^2 f_0(\lambda;z,\tau)+i\lambda f_1(\lambda;z,\tau)+f_2(\lambda;z,\tau)\right)\end{aligned}$$

Applying the time transform to the non-local condition we get

$$\begin{aligned}\int_0^1 K(y) f_0(\lambda; y, \tau) dy &= \int_0^1 K(y) \int_0^t e^{-i\lambda^3 s} q(y,s) ds dy \\ &= \int_0^t e^{-i\lambda^3 s} \int_0^1 K(y) q(y,s) dy ds \\ &= \int_0^t e^{-i\lambda^3 s} g_0(s) ds =: h_0(\lambda)\end{aligned}$$

Evaluating GGR at $z = 1, \tau = t$ we obtain

$$\begin{aligned}\widehat{q_0}(\lambda; y, 1) - e^{-i\lambda^3 t} \widehat{q}(\lambda, t; y, 1) =& e^{-i\lambda y} \left(-\lambda^2 f_0(\lambda; y) + i\lambda f_1(\lambda; y) + f_2(\lambda; y) \right) \\ & - e^{-i\lambda} f_2(\lambda; 1),\end{aligned} \tag{2}$$

since both $f_0(\lambda; 1)$ and $f_1(\lambda; 1)$ evaluate to 0 because of the boundary conditions. Now multiplying the above expression by $e^{i\lambda y} K(y)$ and integration it in y from 0 to 1 we obtain the global relation of non-local type

$$\begin{aligned}& i\lambda \int_0^1 K(y) f_1(\lambda; y) dy + \int_0^1 K(y) f_2(\lambda; y) dy - e^{-i\lambda} f_2(\lambda; 1) \int_0^1 K(y) e^{i\lambda y} dy \\ &= \lambda^2 h_0(\lambda) + \int_0^1 K(y) e^{i\lambda y} \widehat{q_0}(\lambda; y, 1) dy - e^{-i\lambda^3 t} \int_0^1 K(y) e^{i\lambda y} \widehat{q}(\lambda, t; y, 1) dy.\end{aligned}$$

The above equation can be looked at as a linear equation with following three unknowns:

$$\begin{aligned}& x_1(\lambda) := \int_0^1 K(y) f_1(\lambda; y) dy \\ & x_2(\lambda) := \int_0^1 K(y) f_2(\lambda; y) dy \\ & f_2(\lambda; 1)\end{aligned}$$

Also let the RHS of the above equation be our datum that is equal to the $Q(\lambda)$. Note that we can apply the following three maps to the above equation to get three equation with three unknowns since $f_j(\lambda; x, \tau)$ is independent of the choice of λ:

$$\lambda \to \lambda \qquad \lambda \to \alpha\lambda \qquad \lambda \to \alpha^2 \lambda$$

where $\alpha = e^{\frac{2\pi}{3} i}$. We get the following matrix:

$$\begin{pmatrix} i\lambda & 1 & -e^{-i\lambda} \int_0^1 K(y) e^{i\lambda y} dy \\ i\alpha\lambda & 1 & -e^{-i\alpha\lambda} \int_0^1 K(y) e^{i\alpha\lambda y} dy \\ i\alpha^2\lambda & 1 & -e^{-i\alpha^2\lambda} \int_0^1 K(y) e^{i\alpha^2 \lambda y} dy \end{pmatrix} \begin{pmatrix} x_1(\lambda) \\ x_2(\lambda) \\ f_2(\lambda; 1) \end{pmatrix} = \begin{pmatrix} Q(\lambda) \\ Q(\alpha\lambda) \\ Q(\alpha\lambda^2) \end{pmatrix}$$

We find the determinant of the matrix:

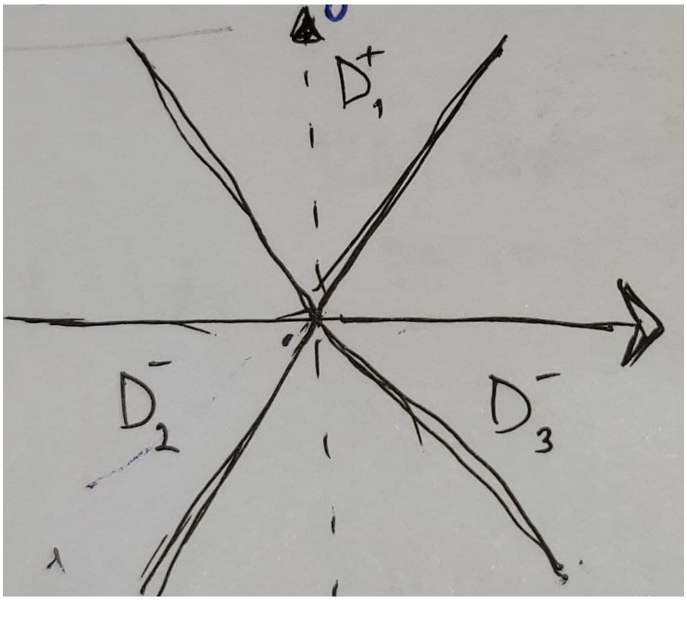
$$\begin{aligned}\Delta(\lambda) = i\lambda\Big[&-e^{-i\alpha\lambda}\int_0^1 K(y)e^{i\alpha\lambda y}dy + \alpha e^{-i\lambda}\int_0^1 K(y)e^{i\lambda y}dy\\ &+e^{-i\alpha^2\lambda}\int_0^1 K(y)e^{i\alpha^2\lambda y}dy - \alpha^2 e^{-i\lambda}\int_0^1 K(y)e^{i\lambda y}d\\ &+\alpha^2 e^{-i\alpha\lambda}\int_0^1 K(y)e^{i\alpha\lambda y}dy - \alpha e^{-i\alpha^2\lambda}\int_0^1 K(y)e^{i\alpha^2\lambda y}dy\Big]\\ = i\lambda(\alpha^2-1)\Big[&e^{-i\alpha\lambda}\int_0^1 K(y)e^{i\alpha\lambda y}dy + \alpha e^{-i\alpha^2\lambda}\int_0^1 K(y)e^{i\alpha^2\lambda y}dy + \alpha^2 e^{-i\lambda}\int_0^1 K(y)e^{i\lambda y}dy\Big]\end{aligned}$$

Now we use Cramer's rule to find one of the unknowns:

$$\begin{aligned}f_2(\lambda;1) =& \frac{i\lambda}{\Delta(\lambda)}\Big[\alpha Q(\alpha^2\lambda) - \alpha^2 Q(\alpha\lambda) - Q(\alpha^2\lambda) + \alpha^2 Q(\lambda) + Q(\alpha\lambda) - \alpha Q(\lambda)\Big]\\ =& \frac{i\lambda}{\Delta(\lambda)}(1-\alpha^2)\Big[\alpha^2 Q(\lambda) + Q(\alpha\lambda) + \alpha Q(\alpha^2\lambda)\Big]\\ =& \frac{-\Big[\alpha^2 Q(\lambda) + Q(\alpha\lambda) + \alpha Q(\alpha^2\lambda)\Big]}{e^{-i\alpha\lambda}\int_0^1 K(y)e^{i\alpha\lambda y}dy + \alpha e^{-i\alpha^2\lambda}\int_0^1 K(y)e^{i\alpha^2\lambda y}dy + \alpha^2 e^{-i\lambda}\int_0^1 K(y)e^{i\lambda y}dy}\end{aligned}$$

Now we need to show the decaying of the following ratio term inside $\int_{\widetilde{\partial D}^-}$ as $\lambda \to \infty$:

$$\frac{-\Big[\alpha^2\int_0^1 K(y)e^{i\lambda y}\widehat{q}(\lambda,t;y,1)dy + \int_0^1 K(y)e^{i\alpha\lambda y}\widehat{q}(\alpha\lambda,t;y,1)dy + \alpha\int_0^1 K(y)e^{i\alpha^2\lambda y}\widehat{q}(\alpha^2\lambda,t;y,1)dy\Big]}{e^{-i\alpha\lambda}\int_0^1 K(y)e^{i\alpha\lambda y}dy + \alpha e^{-i\alpha^2\lambda}\int_0^1 K(y)e^{i\alpha^2\lambda y}dy + \alpha^2 e^{-i\lambda}\int_0^1 K(y)e^{i\lambda y}dy} \tag{3}$$



4 Asymptotic analysis within $cl(\widetilde{D}_2^-)$

Decaying terms: $e^{-i\lambda}, e^{-i\alpha\lambda}, e^{i\alpha^2\lambda}$.
Blowing up terms: $e^{i\lambda}, e^{i\alpha\lambda}, e^{-i\alpha^2\lambda}$.

4.1 Denominator

$$\begin{aligned}\int_0^1 K(y)e^{i\lambda y}dy =&\frac{1}{i\lambda}\Big[K(y)e^{i\lambda y}\Big]_{y=0}^{y=1} - \frac{1}{i\lambda}\int_0^1 K'(y)e^{i\lambda y}dy\\ =&\frac{1}{i\lambda}\Big[K(1)e^{i\lambda}-K(0)\Big] - \frac{1}{(i\lambda)^2}\Big[K'(1)e^{i\lambda}-K'(0)\Big] + \frac{1}{(i\lambda)^2}\int_0^1 K''(y)e^{i\lambda y}dy\\ =&\mathcal{O}\left(\left|\frac{e^{i\lambda}}{\lambda}\right|\right).\end{aligned}$$

Similarly,

$$\int_0^1 K(y)e^{i\alpha\lambda y}dy =\mathcal{O}\left(\left|\frac{e^{i\alpha\lambda}}{\lambda}\right|\right).$$

The last integral term:

$$\begin{aligned}\int_0^1 K(y)e^{i\alpha^2\lambda y}dy =&\frac{1}{i\alpha^2\lambda}\Big[K(y)e^{i\alpha^2\lambda y}\Big]_{y=0}^{y=1} - \frac{1}{i\alpha^2\lambda}\int_0^1 K'(y)e^{i\alpha^2\lambda y}\\ =&\frac{1}{i\alpha^2\lambda}\Big[K(1)e^{i\alpha^2\lambda}-K(0)\Big] - \frac{1}{(i\alpha^2\lambda)^2}\Big[K'(1)e^{i\alpha^2\lambda}-K'(0)\Big] + \frac{1}{(i\alpha^2\lambda)^2}\int_0^1 K''(y)e^{i\alpha^2\lambda y}dy\\ =&\mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right).\end{aligned}$$

Thus, the decay rate of the denominator is

$$\begin{aligned}e^{-i\alpha\lambda}&\int_0^1 K(y)e^{i\alpha\lambda y}dy + \alpha e^{-i\alpha^2\lambda}\int_0^1 K(y)e^{i\alpha^2\lambda y}dy + \alpha^2 e^{-i\lambda}\int_0^1 K(y)e^{i\lambda y}dy\\ &=e^{-i\alpha\lambda}\mathcal{O}\left(\left|\frac{e^{i\alpha\lambda}}{\lambda}\right|\right) + e^{-i\alpha^2\lambda}\mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right) + e^{-i\lambda}\mathcal{O}\left(\left|\frac{e^{i\lambda}}{\lambda}\right|\right)\\ &=\mathcal{O}\left(\left|\frac{e^{-i\alpha^2\lambda}}{\lambda}\right|\right)\end{aligned}$$

4.2 Numerator

4.2.1 First term

Note that

$$\frac{d}{dy}\widehat{q}(\lambda,t;y,1) =\frac{d}{dy}\int_y^1 e^{-i\lambda x}q(x,t)dx = -e^{-i\lambda y}q(y,t)$$

and

$$\begin{aligned}\widehat{q}(\lambda,t;0,1) &= \int_0^1 e^{-i\lambda x} q(x,t)dx \\ &\leq \left|\int_0^1 e^{-i\lambda x} q(x,t)dx\right| \\ &\leq \int_0^1 |e^{-i\lambda x}||q(x,t)|dx \\ &\leq (1-0) \max_{x\in[0,1]} |e^{-i\lambda x}| \max_{x\in[0,1]} |q(x,t)| \\ &\leq \max_{x\in[0,1]} |q(x,t)| \\ &\Longrightarrow \text{bounded}\end{aligned}$$

$$\begin{aligned}\int_0^1 e^{i\lambda y} K(y)\widehat{q}(\lambda,t;y,1)dy =& \frac{1}{i\lambda}\left[e^{i\lambda y}K(y)\widehat{q}(\lambda,t;y,1)\right]_{y=0}^{y=1} - \frac{1}{i\lambda}\int_0^1 e^{i\lambda y}\left[K(y)(-e^{-i\lambda y}q(y,t)) + K'(y)\widehat{q}(\lambda,t;y,1)\right] \\ =& \underbrace{-\frac{1}{i\lambda}K(0)\widehat{q}(\lambda,t;0,1)}_{=\mathcal{O}(|\frac{1}{\lambda}|)} + \underbrace{\frac{1}{i\lambda}\int_0^1 K(y)q(y,t)dy}_{=\mathcal{O}(|\frac{1}{\lambda}|)} - \frac{1}{i\lambda}\int_0^1 e^{i\lambda y}K'(y)\widehat{q}(\lambda,t;y,1)dy \\ =& \mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)\end{aligned}$$

4.2.2 Second term

Similarly, note that

$$\frac{d}{dy}\widehat{q}(\alpha\lambda,t;y,1) = \frac{d}{dy}\int_y^1 e^{-i\alpha\lambda x}q(x,t)dx = -e^{-i\alpha\lambda y}q(y,t)$$

and

$$\begin{aligned}\widehat{q}(\alpha\lambda,t;0,1) &= \int_0^1 e^{-i\alpha\lambda x} q(x,t)dx \\ &\leq \left|\int_0^1 e^{-i\alpha\lambda x} q(x,t)dx\right| \\ &\leq \int_0^1 |e^{-i\alpha\lambda x}||q(x,t)|dx \\ &\leq (1-0) \max_{x\in[0,1]} |e^{-i\alpha\lambda x}| \max_{x\in[0,1]} |q(x,t)| \\ &\leq \max_{x\in[0,1]} |q(x,t)| \\ &\Longrightarrow \text{bounded}\end{aligned}$$

So,

$$
\begin{aligned}
&\int_0^1 e^{i\alpha\lambda y} K(y)\widehat{q}(\alpha\lambda, t; y, 1)dy = \\
&\qquad = \frac{1}{i\alpha\lambda}\left[e^{i\alpha\lambda y} K(y)\widehat{q}(\alpha\lambda, t; y, 1)\right]_{y=0}^{y=1} - \frac{1}{i\alpha\lambda}\int_0^1 e^{i\alpha\lambda y}\left[K(y)(-e^{-i\alpha\lambda y}q(y,t)) + K'(y)\widehat{q}(\alpha\lambda, t; y, 1)\right] dy \\
&\qquad = \underbrace{-\frac{1}{i\alpha\lambda}K(0)\widehat{q}(\alpha\lambda, t; 0, 1)}_{=\mathcal{O}(\left|\frac{1}{\lambda}\right|)} + \underbrace{\frac{1}{i\alpha\lambda}\int_0^1 K(y)q(y,t)dy}_{=\mathcal{O}(\left|\frac{1}{\lambda}\right|)} - \frac{1}{i\alpha\lambda}\int_0^1 e^{i\alpha\lambda y} K'(y)\widehat{q}(\alpha\lambda, t; y, 1)dy \\
&\qquad = \mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)
\end{aligned}
$$

4.2.3 Third term

Note that

$$
\begin{aligned}
\widehat{q}(\alpha^2\lambda, t; 0, 1) &= \int_0^1 e^{-i\alpha^2\lambda x} q(x,t)dx \\
&= -\frac{1}{i\alpha^2\lambda}\left[e^{-i\alpha^2\lambda x} q(x,t)\right]_{x=0}^{x=1} + \frac{1}{i\alpha^2\lambda}\int_0^1 e^{-i\alpha^2\lambda x} q_x(x,t)dx \\
&= -\frac{1}{i\alpha^2\lambda} e^{-i\alpha^2\lambda} q(1,t) + \frac{1}{i\alpha^2\lambda} q(0,t) + \frac{1}{i\alpha^2\lambda}\int_0^1 e^{-i\alpha^2\lambda x} q_x(x,t)dx \\
&= \mathcal{O}\left(\left|\frac{e^{-i\alpha^2\lambda}}{\lambda}\right|\right)
\end{aligned}
$$

So,

$$
\begin{aligned}
&\int_0^1 e^{i\alpha^2\lambda y} K(y)\widehat{q}(\alpha^2\lambda, t; y, 1)dy = \\
&\quad = \frac{1}{i\alpha^2\lambda}\left[e^{i\alpha^2\lambda y} K(y)\widehat{q}(\alpha^2\lambda, t; y, 1)\right]_{y=0}^{y=1} - \frac{1}{i\alpha^2\lambda}\int_0^1 e^{i\alpha^2\lambda y}\left[K(y)(-e^{-i\alpha^2\lambda y}q(y,t)) + K'(y)\widehat{q}(\alpha^2\lambda, t; y, 1)\right] dy \\
&\quad = -\frac{1}{i\alpha^2\lambda}K(0)\widehat{q}(\alpha^2\lambda, t; 0, 1) + \underbrace{\frac{1}{i\alpha^2\lambda}\int_0^1 K(y)q(y,t)dy}_{=\mathcal{O}(\left|\frac{1}{\lambda}\right|)} - \frac{1}{i\alpha^2\lambda}\int_0^1 e^{i\alpha^2\lambda y} K'(y)\widehat{q}(\alpha^2\lambda, t; y, 1)dy \\
&\quad = \mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)\mathcal{O}\left(\left|\frac{e^{-i\alpha^2\lambda}}{\lambda}\right|\right) \\
&\quad = \mathcal{O}\left(\left|\frac{e^{-i\alpha^2\lambda}}{\lambda^2}\right|\right)
\end{aligned}
$$

So the decay rate of the numerator is

$$-\Big[\alpha^2\int_0^1 K(y)e^{i\lambda y}\widehat{q}(\lambda,t;y,1)dy+\int_0^1 K(y)e^{i\alpha\lambda y}\widehat{q}(\alpha\lambda,t;y,1)dy+\alpha\int_0^1 K(y)e^{i\alpha^2\lambda y}\widehat{q}(\alpha^2\lambda,t;y,1)dy\Big]=$$
$$=\mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)+\mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)+\mathcal{O}\left(\left|\frac{1}{\lambda^2e^{i\alpha^2\lambda}}\right|\right)$$
$$=\mathcal{O}\left(\left|\frac{e^{-i\alpha^2\lambda}}{\lambda^2}\right|\right).$$

Thus, the decay rate of the ratio term inside $\int_{\partial\widetilde{D}_2^-}$ is

$$\frac{\mathcal{O}\left(\left|\frac{e^{-i\alpha^2\lambda}}{\lambda^2}\right|\right)}{\mathcal{O}\left(\left|\frac{e^{-i\alpha^2\lambda}}{\lambda}\right|\right)}=\mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)$$

Hence, as $\lambda\to\infty$, this term approaches 0 within $cl(\widetilde{D}_2^-)$.

5 Asymptotic analysis within $cl(\widetilde{D}_3^-)$

Decaying terms: $e^{-i\lambda}, e^{i\alpha\lambda}, e^{-i\alpha^2\lambda}$.
Blowing up terms: $e^{i\lambda}, e^{-i\alpha\lambda}, e^{i\alpha^2\lambda}$.

Note that the decay rate of the ratio term (3), which can is a function of λ is independent of a particular λ mapping. Moreover, by applying a particular mapping $\lambda\to\alpha\lambda$ to the decaying and blowing up terms within $cl(\widetilde{D}_3^-)$ we can retrieve the decaying and the blowing up terms within $cl(\widetilde{D}_2^-)$. These two facts are enough to conclude that the decay rate of the entire ratio term (3) within $cl(\widetilde{D}_3^-)$ is the same as the decay rate within $cl(\widetilde{D}_2^-)$. So, the decay rate of the ratio term inside $\int_{\partial\widetilde{D}_3^-}$ is $\mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)$ Thus, as $\lambda\to\infty$, this term approaches 0 within $cl(\widetilde{D}_3^-)$.

6 Asymptotic analysis within $cl(\widetilde{D}_1^+)$

First, we evaluate GGR (2) at $y=0$ and rearrange it:

$$\widehat{q_0}(\lambda;0,1)-e^{-i\lambda^3t}\widehat{q}(\lambda,t;0,1)+e^{-i\lambda}f_2(\lambda;1)=-\lambda^2f_0(\lambda;0)+i\lambda f_1(\lambda;0)+f_2(\lambda;0)$$

Let us use Σf for the RHS of the above equation. Note that Σf is the term appearing inside $\int_{\partial\widetilde{D}^+}$ in the EFt and we need to show that it decays as $\lambda\to\infty$:

$$\int_{\partial\widetilde{D}^+}e^{i\lambda x+i\lambda^3t}\big[-\lambda^2f_0(\lambda;0)+i\lambda f_1(\lambda;0)+f_2(\lambda;0)\big]d\lambda=$$
$$=\int_{\partial\widetilde{D}^+}e^{i\lambda x+i\lambda^3t}\big[\widehat{q_0}(\lambda;0,1)-e^{-i\lambda^3t}\widehat{q}(\lambda,t;0,1)+e^{-i\lambda}f_2(\lambda;1)\big]d\lambda$$

First of all, $\widehat{q_0}(\lambda;0,1)$ is known, so we can factor it outside of the integral. We only need to show the decaying of $-e^{-i\lambda^3 t}\widehat{q}(\lambda,t;0,1)+e^{-i\lambda}f_2(\lambda;1)$ inside the $\int_{\partial\widetilde{D}^+}$. Factoring out $e^{-i\lambda^3 t}$ and using the ratio term of $f_2(\lambda;1)$ in (3) we get the following ratio term:

$$-\widehat{q}(\lambda,t;0,1) \tag{4}$$
$$+\frac{e^{-i\lambda}\Big[\alpha^2\int_0^1 K(y)e^{i\lambda y}\widehat{q}(\lambda,t;y,1)dy+\int_0^1 K(y)e^{i\alpha\lambda y}\widehat{q}(\alpha\lambda,t;y,1)dy+\alpha\int_0^1 K(y)e^{i\alpha^2\lambda y}\widehat{q}(\alpha^2\lambda,t;y,1)dy\Big]}{e^{-i\alpha\lambda}\int_0^1 K(y)e^{i\alpha\lambda y}dy+\alpha e^{-i\alpha^2\lambda}\int_0^1 K(y)e^{i\alpha^2\lambda y}dy+\alpha^2 e^{-i\lambda}\int_0^1 K(y)e^{i\lambda y}dy}$$
$$=\frac{-\widehat{q}(\lambda,t;0,1)\Big[e^{-i\alpha\lambda}\int_0^1 K(y)e^{i\alpha\lambda y}dy+\alpha e^{-i\alpha^2\lambda}\int_0^1 K(y)e^{i\alpha^2\lambda y}dy+\alpha^2 e^{-i\lambda}\int_0^1 K(y)e^{i\lambda y}dy\Big]+}{}$$
$$\frac{e^{-i\lambda}\Big[\alpha^2\int_0^1 K(y)e^{i\lambda y}\widehat{q}(\lambda,t;y,1)dy+\int_0^1 K(y)e^{i\alpha\lambda y}\widehat{q}(\alpha\lambda,t;y,1)dy+\alpha\int_0^1 K(y)e^{i\alpha^2\lambda y}\widehat{q}(\alpha^2\lambda,t;y,1)dy\Big]}{e^{-i\alpha\lambda}\int_0^1 K(y)e^{i\alpha\lambda y}dy+\alpha e^{-i\alpha^2\lambda}\int_0^1 K(y)e^{i\alpha^2\lambda y}dy+\alpha^2 e^{-i\lambda}\int_0^1 K(y)e^{i\lambda y}dy}$$

Now, we go back to the asymptotic analysis within $cl(\widetilde{D}_1^+)$:
Decaying terms: $e^{i\lambda}, e^{-i\alpha\lambda}, e^{-i\alpha^2\lambda}$.
Blowing up terms: $e^{-i\lambda}, e^{i\alpha\lambda}, e^{i\alpha^2\lambda}$.

A method used in the section on the decay analysis within $cl(\widetilde{D}_2^-)$ will help us find the following decay rates of some of the terms in the above fraction:

$$\begin{aligned}
\int_0^1 K(y)e^{i\lambda y}\widehat{q}(\lambda,t;y,1)dy &= \mathcal{O}\left(\left|\frac{e^{-i\lambda}}{\lambda}\right|\right)\\
\int_0^1 K(y)e^{i\alpha\lambda y}\widehat{q}(\alpha\lambda,t;y,1)dy &= \mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)\\
\int_0^1 K(y)e^{i\alpha^2\lambda y}\widehat{q}(\alpha^2\lambda,t;y,1)dy &= \mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)\\
\int_0^1 K(y)e^{i\alpha\lambda y}dy &= \mathcal{O}\left(\left|\frac{e^{i\alpha\lambda}}{\lambda}\right|\right)\\
\int_0^1 K(y)e^{i\alpha^2\lambda y}dy &= \mathcal{O}\left(\left|\frac{e^{i\alpha^2\lambda}}{\lambda}\right|\right)\\
\int_0^1 K(y)e^{i\lambda y}dy &= \mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right).
\end{aligned}$$

6.1 Denominator

Based on the above the decay rate of the denominator is

$$e^{-i\alpha\lambda}\underbrace{\int_0^1 K(y)e^{i\alpha\lambda y}dy}_{=\mathcal{O}\left(\left|\frac{e^{i\alpha\lambda}}{\lambda}\right|\right)}+\alpha e^{-i\alpha^2\lambda}\underbrace{\int_0^1 K(y)e^{i\alpha^2\lambda y}dy}_{=\mathcal{O}\left(\left|\frac{e^{i\alpha^2\lambda}}{\lambda}\right|\right)}+\alpha^2 e^{-i\lambda}\underbrace{\int_0^1 K(y)e^{i\lambda y}dy}_{=\mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)}=\mathcal{O}\left(\left|\frac{e^{-i\lambda}}{\lambda}\right|\right).$$

6.2 Numerator

First, we examine the decay rate of $\widehat{q}(\lambda, t; 0, 1)$:

$$\begin{aligned}\widehat{q}(\lambda, t; 0, 1) &= \int_0^1 e^{-i\lambda x} q(x,t)dx \\ &= -\frac{1}{i\lambda}\left[e^{-i\lambda x} q(x,t)\right]_{x=0}^{x=1} + \frac{1}{i\lambda}\int_0^1 e^{-i\lambda x} q_x(x,t)dx \\ &= -\frac{1}{i\lambda}e^{-i\lambda} q(1,t) + \frac{1}{i\lambda}q(0,t) + \frac{1}{i\lambda}\int_0^1 e^{-i\lambda x} q_x(x,t)dx \\ &= \mathcal{O}\left(\left|\frac{e^{-i\lambda}}{\lambda}\right|\right)\end{aligned}$$

Let us write out once again the entire numerator:

$$\begin{aligned}&-\underbrace{\widehat{q}(\lambda, t; 0, 1)}_{=\mathcal{O}\left(\left|\frac{e^{-i\lambda}}{\lambda}\right|\right)}\Bigg[\underbrace{e^{-i\alpha\lambda}\int_0^1 K(y)e^{i\alpha\lambda y}dy}_{=\mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)} + \underbrace{\alpha e^{-i\alpha^2\lambda}\int_0^1 K(y)e^{i\alpha^2\lambda y}dy}_{=\mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)} + \underbrace{\alpha^2 e^{-i\lambda}\int_0^1 K(y)e^{i\lambda y}dy}_{=\mathcal{O}\left(\left|\frac{e^{-i\lambda}}{\lambda}\right|\right)}\Bigg] + \\ &+ e^{-i\lambda}\Bigg[\underbrace{\alpha^2\int_0^1 K(y)e^{i\lambda y}\widehat{q}(\lambda, t; y, 1)dy}_{=\mathcal{O}\left(\left|\frac{e^{-i\lambda}}{\lambda}\right|\right)} + \underbrace{\int_0^1 K(y)e^{i\alpha\lambda y}\widehat{q}(\alpha\lambda, t; y, 1)dy}_{=\mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)} + \underbrace{\alpha\int_0^1 K(y)e^{i\alpha^2\lambda y}\widehat{q}(\alpha^2\lambda, t; y, 1)dy}_{=\mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)}\Bigg]\end{aligned}$$

This poses a challenge since there are a number of terms that are blowing up faster or at the same rate as the denominator. We will try to resolve this challenge by grouping some of the problematic terms together and examining them. In order to make this clearer we will be removing all the non-problematic terms so that at every step we have less and less terms to deal with. A term is considered non-problematic if it has a decay rate higher than the denominator, which has a blow up rate of $\mathcal{O}\left(\left|\frac{e^{-i\lambda}}{\lambda}\right|\right)$. If we manage to remove all the terms that means that we have shown that the entire numerator decays faster than the denominator. The first non-problematic term is

$$-\underbrace{\widehat{q}(\lambda, t; 0, 1)}_{=\mathcal{O}\left(\left|\frac{e^{-i\lambda}}{\lambda}\right|\right)}\Bigg[\underbrace{e^{-i\alpha\lambda}\int_0^1 K(y)e^{i\alpha\lambda y}dy}_{=\mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)} + \underbrace{\alpha e^{-i\alpha^2\lambda}\int_0^1 K(y)e^{i\alpha^2\lambda y}dy}_{=\mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)}\Bigg]$$

since it has a decay rate of $\mathcal{O}\left(\left|\frac{e^{-i\lambda}}{\lambda^2}\right|\right)$. Hence, we are left with the following problematic terms:

$$- \underbrace{\widehat{q}(\lambda,t;0,1)}_{=\mathcal{O}\left(\left|\frac{e^{-i\lambda}}{\lambda}\right|\right)}\Big[\underbrace{\alpha^2 e^{-i\lambda}\int_0^1 K(y)e^{i\lambda y}dy}_{=\mathcal{O}\left(\left|\frac{e^{-i\lambda}}{\lambda}\right|\right)}\Big]+ \tag{5}$$

$$+\, e^{-i\lambda}\Big[\underbrace{\alpha^2\int_0^1 K(y)e^{i\lambda y}\widehat{q}(\lambda,t;y,1)dy}_{=\mathcal{O}\left(\left|\frac{e^{-i\lambda}}{\lambda}\right|\right)}+\underbrace{\int_0^1 K(y)e^{i\alpha\lambda y}\widehat{q}(\alpha\lambda,t;y,1)dy}_{=\mathcal{O}(|\frac{1}{\lambda}|)}+\underbrace{\alpha\int_0^1 K(y)e^{i\alpha^2\lambda y}\widehat{q}(\alpha^2\lambda,t;y,1)dy}_{=\mathcal{O}(|\frac{1}{\lambda}|)}\Big]$$

Let us now simplify and examine $-\widehat{q}(\lambda,t;0,1)e^{-i\lambda}\int_0^1 K(y)e^{i\lambda y}dy+e^{-i\lambda}\int_0^1 K(y)e^{i\lambda y}\widehat{q}(\lambda,t;y,1)dy$ which without any further work has a blow up rate of $\mathcal{O}\left(\left|\frac{(e^{-i\lambda})^2}{\lambda}\right|\right)$:

$$\begin{aligned}
&-\widehat{q}(\lambda,t;0,1)e^{-i\lambda}\int_0^1 K(y)e^{i\lambda y}dy + e^{-i\lambda}\int_0^1 K(y)e^{i\lambda y}\widehat{q}(\lambda,t;y,1)dy\\
&=-\,e^{-i\lambda}\left(\widehat{q}(\lambda,t;0,1)\int_0^1 K(y)e^{i\lambda y}dy-\int_0^1 K(y)e^{i\lambda y}\widehat{q}(\lambda,t;y,1)dy\right)\\
&=-\,e^{-i\lambda}\int_0^1 K(y)e^{i\lambda y}\left(\widehat{q}(\lambda,t;0,1)-\widehat{q}(\lambda,t;y,1)\right)dy\\
&=-\,e^{-i\lambda}\int_0^1 e^{i\lambda y}K(y)\widehat{q}(\lambda,t;0,y)dy
\end{aligned}$$

Hence, equation (5) with all problematic terms simplifies to

$$-e^{-i\lambda}\int_0^1 e^{i\lambda y}K(y)\widehat{q}(\lambda,t;0,y)dy+e^{-i\lambda}\int_0^1 K(y)e^{i\alpha\lambda y}\widehat{q}(\alpha\lambda,t;y,1)dy+e^{-i\lambda}\alpha\int_0^1 K(y)e^{i\alpha^2\lambda y}\widehat{q}(\alpha^2\lambda,t;y,1)dy \tag{6}$$

Let us examine the first term in the above equation:

$$\begin{aligned}
&-\,e^{-i\lambda}\int_0^1 e^{i\lambda y}K(y)\widehat{q}(\lambda,t;0,y)dy\\
&=-e^{-i\lambda}\left(\frac{1}{i\lambda}\left[e^{i\lambda y}K(y)\widehat{q}(\lambda,t;0,y)\right]_{y=0}^{y=1}-\frac{1}{i\lambda}\int_0^1 e^{i\lambda y}\left[K(y)e^{-i\lambda y}q(y,t)+K'(y)\widehat{q}(\lambda,t;0,y)\right]dy\right)\\
&=-e^{-i\lambda}\left(\frac{1}{i\lambda}e^{i\lambda}K(1)\widehat{q}(\lambda,t;0,1)-\frac{1}{i\lambda}\int_0^1 K(y)q(y,t)dy-\frac{1}{i\lambda}\int_0^1 e^{i\lambda y}K'(y)\widehat{q}(\lambda,t;0,y)dy\right)\\
&=-e^{-i\lambda}\left(\underbrace{\frac{1}{i\lambda}e^{i\lambda}K(1)\widehat{q}(\lambda,t;0,1)}_{=\mathcal{O}\left(\left|\frac{1}{\lambda^2}\right|\right)}-\underbrace{\frac{1}{i\lambda}g_0(t)}_{=\mathcal{O}(|\frac{1}{\lambda}|)}-\frac{1}{i\lambda}\int_0^1 e^{i\lambda y}K'(y)\widehat{q}(\lambda,t;0,y)dy\right)
\end{aligned}$$

The problematic term in the equation above is $e^{-i\lambda} \times \frac{1}{i\lambda} g_0(t)$. However, since $g_0(t)$ is a known function given as a non-local condition, we can factor out this term outside of the integral. Now we are left with the second and third integral terms in (6). First we examine the decay rate of $\widehat{q}(\alpha\lambda, t; 0, 1)$ and $\widehat{q}(\alpha^2\lambda, t; 0, 1)$:

$$\begin{aligned}
\widehat{q}(\alpha\lambda, t; 0, 1) =& \int_0^1 e^{-i\alpha\lambda x} q(x,t)dx \\
=& -\frac{1}{\alpha\lambda}\left[e^{-i\alpha\lambda x} q(x,t)\right]_{x=0}^{x=1} + \frac{1}{\alpha\lambda}\int_0^1 e^{-i\alpha\lambda x} q_x(x,t)dx \\
=& -\frac{1}{i\alpha\lambda} e^{-i\alpha\lambda} q(1,t) + \frac{1}{i\alpha\lambda} q(0,t) + \frac{1}{i\alpha\lambda}\int_0^1 e^{-i\alpha\lambda x} q_x(x,t)dx \\
=& \mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right) \\
\widehat{q}(\alpha^2\lambda, t; 0, 1) =& \int_0^1 e^{-i\alpha^2\lambda x} q(x,t)dx \\
=& -\frac{1}{\alpha^2\lambda}\left[e^{-i\alpha^2\lambda x} q(x,t)\right]_{x=0}^{x=1} + \frac{1}{\alpha^2\lambda}\int_0^1 e^{-i\alpha^2\lambda x} q_x(x,t)dx \\
=& -\frac{1}{i\alpha^2\lambda} e^{-i\alpha^2\lambda} q(1,t) + \frac{1}{i\alpha^2\lambda} q(0,t) + \frac{1}{i\alpha^2\lambda}\int_0^1 e^{-i\alpha^2\lambda x} q_x(x,t)dx \\
=& \mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)
\end{aligned}$$

Also remember that the second and third integral terms in (6) can be expanded using integration by parts as follows:

$$\begin{aligned}
& e^{-i\lambda}\int_0^1 e^{i\alpha\lambda y} K(y)\widehat{q}(\alpha\lambda, t; y, 1)dy = \\
&= e^{-i\lambda}\left(-\frac{1}{i\alpha\lambda}K(0)\widehat{q}(\alpha\lambda, t; 0, 1) + \frac{1}{i\alpha\lambda}\int_0^1 K(y)q(y,t)dy - \frac{1}{i\alpha\lambda}\int_0^1 e^{i\alpha\lambda y} K'(y)\widehat{q}(\alpha\lambda, t; y, 1)dy\right) \\
&= e^{-i\lambda}\left(\underbrace{-\frac{1}{i\alpha\lambda}K(0)\widehat{q}(\alpha\lambda, t; 0, 1)}_{=\mathcal{O}\left(\left|\frac{1}{\lambda^2}\right|\right)} + \underbrace{\frac{1}{i\alpha\lambda} g_0(t)}_{=\mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)} - \frac{1}{i\alpha\lambda}\int_0^1 e^{i\alpha\lambda y} K'(y)\widehat{q}(\alpha\lambda, t; y, 1)dy\right)
\end{aligned}$$

where the problematic term is $e^{-i\lambda} \times \frac{1}{i\alpha\lambda} g_0(t)$, which can be factored out since $g_0(t)$ is known. And

$$e^{-i\lambda}\int_0^1 e^{i\alpha^2\lambda y}K(y)\widehat{q}(\alpha^2\lambda,t;y,1)dy =$$

$$= e^{-i\lambda}\left(-\frac{1}{i\alpha^2\lambda}K(0)\widehat{q}(\alpha^2\lambda,t;0,1)+\frac{1}{i\alpha^2\lambda}\int_0^1 K(y)q(y,t)dy-\frac{1}{i\alpha^2\lambda}\int_0^1 e^{i\alpha^2\lambda y}K'(y)\widehat{q}(\alpha^2\lambda,t;y,1)dy\right)$$

$$= e^{-i\lambda}\left(\underbrace{-\frac{1}{i\alpha^2\lambda}K(0)\widehat{q}(\alpha^2\lambda,t;0,1)}_{=\mathcal{O}\left(\left|\frac{1}{\lambda^2}\right|\right)}+\underbrace{\frac{1}{i\alpha^2\lambda}g_0(t)}_{=\mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)}-\frac{1}{i\alpha^2\lambda}\int_0^1 e^{i\alpha^2\lambda y}K'(y)\widehat{q}(\alpha^2\lambda,t;y,1)dy\right)$$

where the problematic term is $e^{-i\lambda}\times\frac{1}{i\alpha^2\lambda},g_0(t)$, which can also be factored out since $g_0(t)$ is known. Hence we have resolved every single problematic term in (5). Thus, the decay rate of the numerator is greater than the decay rate of the denominator. To be more precise, the decay rate of the entire ratio term (4) is

$$\frac{\mathcal{O}\left(\left|\frac{e^{-i\lambda}}{\lambda^2}\right|\right)}{\mathcal{O}\left(\left|\frac{e^{-i\lambda}}{\lambda}\right|\right)}=\mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)$$

Hence, as $\lambda\to\infty$, the ratio term (4) approaches 0 within $cl(\widetilde{D}_1^+)$.