Contents

Problem 2. Linearised KdV

Bekzod Normatov

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1 Problem set up

Consider the following Linearised KdV initial non-local value problem:

$$\begin{aligned}
q_t + q_{xxx} =& 0 & (x,t) \in& (0,L) \times (0,T), & (PDE)\\
q(0,t) =& 0 & t \in& [0,T], & (BC1)\\
q(1,t) =& 0 & t \in& [0,T], & (BC2)\\
\int_0^1 K(y) q(y,t) dy =& g_o(t) & t \in& [0,T], & (NLC)\\
q(x,0) =& q_0(x) & t \in& [0,1], & (IC)
\end{aligned}$$

where q_0, g_0 are known smooth functions.

We use $\widehat{\phi}$ to represent the Fourier transform of $\phi \in C^{\infty}[0,1]$:

$$\widehat{\phi}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} \phi(x) dx$$

Use the following notation simplifications:

$$\begin{aligned}
f_j(\lambda; x, \tau) &= \int_0^{\tau} e^{-i\lambda^3 s} \partial_X^j q(x,s) ds\\
\widehat{q_0}(\lambda; y, z) &= \int_y^z e^{-i\lambda \xi} q_0(\xi) d\xi\\
\widehat{q}(\lambda, \tau; y, z) &= \int_y^z e^{-i\lambda \xi} q(\xi, \tau) d\xi.
\end{aligned}$$

2 Stage 1

Assume there exists $q : [0,1] \times [0,T] \to \mathbb{C}$, as smooth as we need, satisfying (PDE) and (IC). Apply Fourier transform to (PDE):

$$\begin{aligned} 0 =& \widehat{q_t + q_{xxx}}(\lambda;t) \\ =& \widehat{q_t}(\lambda;t) + \widehat{q_{xxx}}(\lambda;t) \\ =& \partial_t \widehat{q}(\lambda;t) - i\lambda^3 \widehat{q}(\lambda;t) + e^{-i\lambda}\Big[\partial_{xx} q(1,t) + i\lambda \partial_x q(1,t) - \lambda^2 q(1,t)\Big] - \Big[\partial_{xx} q(0,t) + i\lambda\partial_x q(0,t) - \lambda^2 q(0,t)\Big]. \end{aligned}$$

Rearrange and multiply by $e^{-i\lambda^3 t}$

$$\partial_t[e^{-i\lambda^3 t}\widehat{q}(\lambda;t)] = e^{-i\lambda^3 t}\Big[\partial_{xx} q(0,t) + i\lambda\partial_x q(0,t) - \lambda^2 q(0,t)\Big] - e^{-i\lambda - i\lambda^3 t}\Big[\partial_{xx} q(1,t) + i\lambda\partial_x q(1,t) - \lambda^2 q(1,t)\Big]$$

Integrate in time to solve the ODE for $\widehat{q}(\lambda;\cdot)$

$$\begin{aligned} e^{-i\lambda^3 t}\widehat{q}(\lambda;t) - \widehat{q}(\lambda;0) = & \int_0^t e^{-i\lambda^3 s}\Big[\partial_{xx} q(0,s) + i\lambda\partial_x q(0,s) - \lambda^2 q(0,s)\Big] ds \\ & - e^{-i\lambda}\int_0^t e^{-i\lambda^3 s}\Big[\partial_{xx} q(1,s) + i\lambda\partial_x q(1,s) - \lambda^2 q(1,s)\Big] ds \end{aligned}$$

Note that (IC) implies $\widehat{q}(\lambda;0) = \widehat{q_0}(\lambda)$. Also introduce notation

$$f_j(\lambda;X,t) := \int_0^t e^{-i\lambda^3 s}\partial_x^j q(X,s) ds$$

Then,

$$e^{-i\lambda^3 t}\widehat{q}(\lambda;t) - \widehat{q_0}(\lambda) = f_2(\lambda;0,t) + i\lambda f_1(\lambda;0,t) - \lambda^2 f_0(\lambda;0,t) - e^{-i\lambda}\big[f_2(\lambda;1,t) + i\lambda f_1(\lambda;1,t) - \lambda^2 f_0(\lambda;1,t)\big].$$

This is Global Relation (GR) valid $\forall t \in [0,T], \forall \lambda \in \mathbb{C}$. Now solve for $q(x,t)$. Rearranging

$$\begin{aligned} \widehat{q}(\lambda;t) =& e^{i\lambda^3 t}\widehat{q_0}(\lambda) + e^{i\lambda^3 t}\big[f_2(\lambda;0,t) + i\lambda f_1(\lambda;0,t) - \lambda^2 f_0(\lambda;0,t)\big] \\ & - e^{-i\lambda + i\lambda^3 t}\big[f_2(\lambda;1,t) + i\lambda f_1(\lambda;1,t) - \lambda^2 f_0(\lambda;1,t)\big] \end{aligned}$$

Applying inverse Fourier transform

$$\begin{aligned} 2\pi q(x,t) = & \int_{-\infty}^{\infty} e^{i\lambda x + i\lambda^3 t}\widehat{q_0}(\lambda) d\lambda + \int_{-\infty}^{\infty} e^{i\lambda x + i\lambda^3 t}\big[f_2(\lambda;0,t) + i\lambda f_1(\lambda;0,t) - \lambda^2 f_0(\lambda;0,t)\big] d\lambda \\ & - \int_{-\infty}^{\infty} e^{i\lambda(x-1) + i\lambda^3 t}\big[f_2(\lambda;1,t) + i\lambda f_1(\lambda;1,t) - \lambda^2 f_0(\lambda;1,t)\big] d\lambda \end{aligned}$$

valid $\forall x \in (0,1), \forall t \in [0,T]$.

We aim to deform the latter two contours of integration away from $\mathbb{R}$. We need the following definitions:

$$\begin{aligned}
\mathbb{C}^{\pm} &:= \{\lambda \in \mathbb{C} : \pm\mathrm{Im}(\lambda) > 0\} \\
D &:= \{\lambda \in \mathbb{C} : \mathrm{Re}(-i\lambda^3) < 0\} \\
D^{\pm} &:= D \cap \mathbb{C}^{\pm} \\
E &:= \{\lambda \in \mathbb{C} : \mathrm{Re}(-i\lambda^3) > 0\} \\
E^{\pm} &:= E \cap \mathbb{C}^{\pm}
\end{aligned}$$

Orient the boundaries of these (unions of) sectors positively; the sector lies to the left of its boundary. Consider $\lambda \in$ clos(E). We want to show that

$$\int_{\partial E^+} e^{i\lambda x + i\lambda^3 t}\big[f_2(\lambda;0,t) + i\lambda f_1(\lambda;0,t) - \lambda^2 f_0(\lambda;0,t)\big]d\lambda = 0$$

First we want to show that

$$\lim_{\lambda\to\infty} e^{i\lambda^3 t}\big[f_2(\lambda;0,t) + i\lambda f_1(\lambda;0,t) - \lambda^2 f_0(\lambda;0,t)\big] = 0$$

Consider

$$\begin{aligned}
\lim_{\lambda\to\infty} e^{i\lambda^3 t} f_j(\lambda;X,t) &= \int_0^t e^{-i\lambda^3(s-t)}\partial_x^j q(X,s)ds \\
&= (-i\lambda)^{-3}\Big[\partial_x^j q(X,t) \underbrace{-e^{i\lambda^3 t}\partial_x^j q(X,0)}_{\text{decays as } \lambda\to\infty} - \int_0^t e^{-i\lambda^3(s-t)}\partial_t\partial_x^j q(X,s)ds\Big]
\end{aligned}$$

Consider

$$\begin{aligned}
\Big|\int_0^t e^{-i\lambda^3(s-t)}\partial_t\partial_x^j q(X,s)ds\Big| &\le \int_0^t \Big|e^{-i\lambda^3(s-t)}\Big|\Big|\partial_t\partial_x^j q(X,s)\Big|ds \\
&\le (t-0)\max_{s\in[0,t]}\big|e^{-i\lambda^3(s-t)}\big| \max_{s\in[0,t]}\big|\partial_t\partial_x^j q(X,s)\big| \\
&\le t\max_{s\in[0,t]}\big|\partial_t\partial_x^j q(X,s)\big| \qquad \forall\lambda \in clos(E)
\end{aligned}$$

So, $e^{i\lambda^3 t} f_j(\lambda;X,t) = \mathcal{O}(|\lambda^{-3}|)$, uniformly in $\arg(\lambda)$ as $\lambda\to\infty$ within clos(E). Similarly, $i\lambda e^{i\lambda^3 t} f_j(\lambda;X,t) = \mathcal{O}(|\lambda^{-2}|)$ and $\lambda^2 e^{i\lambda^3 t} f_j(\lambda;X,t) = \mathcal{O}(|\lambda^{-1}|)$.

Then, $e^{i\lambda^3 t}\big[f_2(\lambda;X,t)+i\lambda f_1(\lambda;X,t)-\lambda^2 f_0(\lambda;X,t)\big] = \mathcal{O}(|\lambda^{-1}|)$, uniformly in $\arg(\lambda)$ as $\lambda\to\infty$ within clos(E). So, we can use Jordan's Lemma. Also, note that $e^{i\lambda^3 t}\big[f_2(\lambda;X,t) + i\lambda f_1(\lambda;X,t) - \lambda^2 f_0(\lambda;X,t)\big]$ is entire by Morera's theorem. So, we can use Cauchy's theorem. Hence, by Jordan's lemma and Cauchy's theorem,

$$\int_{\partial E^+} e^{i\lambda x + i\lambda^3 t}\big[f_2(\lambda;0,t) + i\lambda f_1(\lambda;0,t) - \lambda^2 f_0(\lambda;0,t)\big]d\lambda = 0$$

So,

$$
\begin{aligned}
&\int_{-\infty}^{\infty} e^{i\lambda x+i\lambda^3 t}\big[f_2(\lambda;0,t)+i\lambda f_1(\lambda;0,t)-\lambda^2 f_0(\lambda;0,t)\big]d\lambda \\
=&\left\{\int_{-\infty}^{\infty}-\int_{\partial E^+}\right\}e^{i\lambda x+i\lambda^3 t}\big[f_2(\lambda;0,t)+i\lambda f_1(\lambda;0,t)-\lambda^2 f_0(\lambda;0,t)\big]d\lambda \\
=&\int_{\partial D^+} e^{i\lambda x+i\lambda^3 t}\big[f_2(\lambda;0,t)+i\lambda f_1(\lambda;0,t)-\lambda^2 f_0(\lambda;0,t)\big]d\lambda.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&-\int_{-\infty}^{\infty} e^{i\lambda(x-1)+i\lambda^3 t}\big[f_2(\lambda;1,t)+i\lambda f_1(\lambda;1,t)-\lambda^2 f_0(\lambda;1,t)\big]d\lambda \\
=&\left\{\int_{\infty}^{-\infty}-\int_{\partial E^-}\right\}e^{i\lambda(x-1)+i\lambda^3 t}\big[f_2(\lambda;1,t)+i\lambda f_1(\lambda;1,t)-\lambda^2 f_0(\lambda;1,t)\big]d\lambda \\
=&\int_{\partial D^-} e^{i\lambda(x-1)+i\lambda^3 t}\big[f_2(\lambda;1,t)+i\lambda f_1(\lambda;1,t)-\lambda^2 f_0(\lambda;1,t)\big]d\lambda.
\end{aligned}
$$

We have arrive at the Ehrenpreis form:

$$
\begin{aligned}
2\pi q(x,t)=&\int_{-\infty}^{\infty} e^{i\lambda x+i\lambda^3 t}\widehat{q_0}(\lambda)d\lambda+\int_{\partial D^+} e^{i\lambda x+i\lambda^3 t}\big[f_2(\lambda;0,t)+i\lambda f_1(\lambda;0,t)-\lambda^2 f_0(\lambda;0,t)\big]d\lambda \\
&+\int_{\partial D^-} e^{i\lambda(x-1)+i\lambda^3 t}\big[f_2(\lambda;1,t)+i\lambda f_1(\lambda;1,t)-\lambda^2 f_0(\lambda;1,t)\big]d\lambda
\end{aligned}
\tag{1}
$$

valid $\forall x\in(0,1), \forall t\in[0,T]$.

3 Stage 2

Assume there exists q that satisfies not only (PDE) and (IC) but also (BC). Remember that the generalized global relation(GGR) is the same for non-local initial value problems as for the boundary value problems that have the same PDE. Hence, we already know the GGR:

$$
\begin{aligned}
\widehat{q_0}(\lambda;y,z)-e^{-i\lambda^3 t}\widehat{q}(\lambda,\tau;y,z)=&e^{-i\lambda y}\left(-\lambda^2 f_0(\lambda;y,\tau)+i\lambda f_1(\lambda;y,\tau)+f_2(\lambda;y,\tau)\right) \\
&-e^{-i\lambda z}\left(-\lambda^2 f_0(\lambda;z,\tau)+i\lambda f_1(\lambda;z,\tau)+f_2(\lambda;z,\tau)\right)
\end{aligned}
\tag{2}
$$

Applying the time transform to the non-local condition we get

$$\begin{aligned}\int_0^1 K(y) f_0(\lambda; y, \tau) dy &= \int_0^1 K(y) \int_0^t e^{-i\lambda^3 s} q(y, s) ds dy \\ &= \int_0^t e^{-i\lambda^3 s} \int_0^1 K(y) q(y, s) dy ds \\ &= \int_0^t e^{-i\lambda^3 s} g_0(s) ds =: h_0(\lambda)\end{aligned}$$

Evaluating GGR (2) at $z = 1, \tau = t$ we obtain

$$\begin{aligned}\widehat{q_0}(\lambda; y, 1) - e^{-i\lambda^3 t}\widehat{q}(\lambda, t; y, 1) =& e^{-i\lambda y}\left(-\lambda^2 f_0(\lambda; y) + i\lambda f_1(\lambda; y) + f_2(\lambda; y)\right) \\ &- e^{-i\lambda}\left(i\lambda f_1(\lambda; 1) + f_2(\lambda; 1)\right),\end{aligned} \tag{3}$$

since $f_0(\lambda; 1)$ evaluates to 0 because of the boundary condition 2. Now multiplying the above expression by $e^{i\lambda y} K(y)$ and integration it in y from 0 to 1 we obtain the following global relation of non-local type

$$\begin{aligned}i\lambda \int_0^1 K(y) f_1(\lambda; y) dy + \int_0^1 K(y) f_2(\lambda; y) dy - e^{-i\lambda} i\lambda f_1(\lambda; 1) \int_0^1 K(y) e^{i\lambda y} dy - e^{-i\lambda} f_2(\lambda; 1) \int_0^1 K(y) e^{i\lambda y} dy \\ = \lambda^2 h_0(\lambda) + \int_0^1 K(y) e^{i\lambda y} \widehat{q_0}(\lambda; y, 1) dy - e^{-i\lambda^3 t} \int_0^1 K(y) e^{i\lambda y} \widehat{q}(\lambda, t; y, 1) dy.\end{aligned} \tag{4}$$

The above equation can be looked at as a linear equation with following four unknowns:

1. $x_1(\lambda) := \int_0^1 K(y) f_1(\lambda; y) dy$
2. $x_2(\lambda) := \int_0^1 K(y) f_2(\lambda; y) dy$
3. $f_1(\lambda; 1)$
4. $f_2(\lambda; 1)$

Also let the RHS of the above equation be our datum that is equal to the $Q(\lambda)$. Now we evaluate GGR (2) at $y = 0, z = 1, \tau = t$ to get

$$i\lambda f_1(\lambda; 0) + f_2(\lambda; 0) - e^{-i\lambda} i\lambda f_1(\lambda; 1) - e^{-i\lambda} f_2(\lambda; 1) = \widehat{q_0}(\lambda; 0, 1) - e^{-i\lambda^3 t}\widehat{q}(\lambda, t; 0, 1) \tag{5}$$

since $f_0(\lambda; 1)$ and $f_0(\lambda; 0)$ evaluate to 0 because of the boundary conditions. The above equation can be looked at as a linear equation with following four unknowns:

1. $f_1(\lambda; 0)$
2. $f_2(\lambda; 0)$
3. $f_1(\lambda; 1)$

4. $f_2(\lambda;1)$

Also let the RHS of the above equation be our datum that is equal to the $N(\lambda)$. Note that we can apply the following three maps to the equations (4) and (5) to get six equations with six unknowns since $f_j(\lambda;x,\tau)$ is independent of the choice of λ:

$$\lambda \to \lambda \qquad\qquad \lambda \to \alpha\lambda \qquad\qquad \lambda \to \alpha^2\lambda$$

where $\alpha = e^{\frac{2\pi}{3}i}$. We get the following matrix:

$$\begin{pmatrix} 1 & i\lambda & -e^{-i\lambda}i\lambda & -e^{-i\lambda} & 0 & 0 \\ 1 & i\alpha\lambda & -e^{-i\alpha\lambda}i\alpha\lambda & -e^{-i\alpha\lambda} & 0 & 0 \\ 1 & i\alpha^2\lambda & -e^{-i\alpha^2\lambda}i\alpha^2\lambda & -e^{-i\alpha^2\lambda} & 0 & 0 \\ 0 & 0 & -e^{-i\lambda}i\lambda\int_0^1 K(y)e^{i\lambda y}dy & -e^{-i\lambda}\int_0^1 K(y)e^{i\lambda y}dy & i\lambda & 1 \\ 0 & 0 & -e^{-i\alpha\lambda}i\alpha\lambda\int_0^1 K(y)e^{i\alpha\lambda y}dy & -e^{-i\alpha\lambda}\int_0^1 K(y)e^{i\alpha\lambda y}dy & i\alpha\lambda & 1 \\ 0 & 0 & -e^{-i\alpha^2\lambda}i\alpha^2\lambda\int_0^1 K(y)e^{i\alpha^2\lambda y}dy & -e^{-i\alpha^2\lambda}\int_0^1 K(y)e^{i\alpha^2\lambda y}dy & i\alpha^2\lambda & 1 \end{pmatrix} \begin{pmatrix} f_2(\lambda;0) \\ f_1(\lambda;0) \\ f_1(\lambda;1) \\ f_2(\lambda;1) \\ x_1(\lambda) \\ x_2(\lambda) \end{pmatrix} = \begin{pmatrix} N(\lambda) \\ N(\alpha\lambda) \\ N(\alpha\lambda^2) \\ Q(\lambda) \\ Q(\alpha\lambda) \\ Q(\alpha\lambda^2) \end{pmatrix}$$

We use the following notational simplifications to make calculations easier:

$$E(\lambda) = e^{-i\lambda} \qquad\qquad \widehat{K}(-\lambda) = \int_0^1 K(y)e^{i\lambda y}dy$$

Now the matrix becomes:

$$\begin{pmatrix} 1 & 1 & E(\lambda) & E(\lambda) & 0 & 0 \\ 1 & \alpha & \alpha E(\alpha\lambda) & E(\alpha\lambda) & 0 & 0 \\ 1 & \alpha^2 & \alpha^2 E(\alpha^2\lambda) & E(\alpha^2\lambda) & 0 & 0 \\ 0 & 0 & E(\lambda)\widehat{K}(-\lambda) & E(\lambda)\widehat{K}(-\lambda) & 1 & 1 \\ 0 & 0 & \alpha E(\alpha\lambda)\widehat{K}(-\alpha\lambda) & (\alpha\lambda)\widehat{K}(-\alpha\lambda) & \alpha & 1 \\ 0 & 0 & \alpha^2 E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda) & E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda) & \alpha^2 & 1 \end{pmatrix} \begin{pmatrix} f_2(\lambda;0) \\ i\lambda f_1(\lambda;0) \\ -i\lambda f_1(\lambda;1) \\ -f_2(\lambda;1) \\ x_1(\lambda) \\ x_2(\lambda) \end{pmatrix} = \begin{pmatrix} N(\lambda) \\ N(\alpha\lambda) \\ N(\alpha\lambda^2) \\ Q(\lambda) \\ Q(\alpha\lambda) \\ Q(\alpha\lambda^2) \end{pmatrix}$$

Rearranging further and simplifying the matrix:

$$\begin{pmatrix} 1 & 1 & E(\lambda) & E(\lambda) & 0 & 0 \\ 0 & \alpha-1 & \alpha E(\alpha\lambda)-E(\lambda) & E(\alpha\lambda)-E(\lambda) & 0 & 0 \\ 0 & \alpha^2-1 & \alpha^2 E(\alpha^2\lambda)-E(\lambda) & E(\alpha^2\lambda)-E(\lambda) & 0 & 0 \\ 0 & 0 & \alpha E(\alpha\lambda)\widehat{K}(-\alpha\lambda)-E(\lambda)\widehat{K}(-\lambda) & E(\alpha\lambda)\widehat{K}(-\alpha\lambda)-E(\lambda)\widehat{K}(-\lambda) & \alpha-1 & 0 \\ 0 & 0 & \alpha^2 E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda)-E(\lambda)\widehat{K}(-\lambda) & E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda)-E(\lambda)\widehat{K}(-\lambda) & \alpha^2-1 & 0 \\ 0 & 0 & E(\lambda)\widehat{K}(-\lambda) & E(\lambda)\widehat{K}(-\lambda) & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} f_2(\lambda;0) \\ i\lambda f_1(\lambda;0) \\ -i\lambda f_1(\lambda;1) \\ -f_2(\lambda;1) \\ x_1(\lambda) \\ x_2(\lambda) \end{pmatrix} = \begin{pmatrix} N(\lambda) \\ N(\alpha\lambda)-N(\lambda) \\ N(\alpha\lambda^2)-N(\lambda) \\ Q(\alpha\lambda)-Q(\lambda) \\ Q(\alpha\lambda^2)-Q(\lambda) \\ Q(\lambda) \end{pmatrix}$$

$$\begin{aligned}
a =&\alpha E(\alpha\lambda) - E(\lambda)\\
A =&E(\alpha\lambda) - E(\lambda)\\
b =&\alpha^2 E(\alpha^2\lambda) - E(\lambda)\\
B =&E(\alpha^2\lambda) - E(\lambda)\\
c =&\alpha E(\alpha\lambda)\widehat{K}(-\alpha\lambda) - E(\lambda)\widehat{K}(-\lambda)\\
C =&E(\alpha\lambda)\widehat{K}(-\alpha\lambda) - E(\lambda)\widehat{K}(-\lambda)\\
d =&\alpha^2 E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda) - E(\lambda)\widehat{K}(-\lambda)\\
D =&E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda) - E(\lambda)\widehat{K}(-\lambda)
\end{aligned}$$

We find the determinant of the matrix:

$$\begin{aligned}
\Delta(\lambda) =&(\alpha-1)(\alpha-1)\Big[(\alpha^2 E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda) - E(\lambda)\widehat{K}(-\lambda)) \times (E(\alpha^2\lambda) - E(\lambda))\\
&\quad - \big(E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda) - E(\lambda)\widehat{K}(-\lambda)\big) \times (\alpha^2 E(\alpha^2\lambda) - E(\lambda))\Big]\\
&+(\alpha-1)(\alpha^2-1)\Big[\big(E(\alpha\lambda)\widehat{K}(-\alpha\lambda) - E(\lambda)\widehat{K}(-\lambda)\big) \times \big(\alpha^2 E(\alpha^2\lambda) - E(\lambda)\big)\\
&\quad - \big(\alpha E(\alpha\lambda)\widehat{K}(-\alpha\lambda) - E(\lambda)\widehat{K}(-\lambda)\big) \times \big(E(\alpha^2\lambda) - E(\lambda)\big)\Big]\\
&+(\alpha^2-1)(\alpha-1)\Big[\big(E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda) - E(\lambda)\widehat{K}(-\lambda)\big) \times \big(\alpha E(\alpha\lambda) - E(\lambda)\big)\\
&\quad - \big(\alpha^2 E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda) - E(\lambda)\widehat{K}(-\lambda)\big) \times \big(E(\alpha\lambda) - E(\lambda)\big)\Big]\\
&+(\alpha^2-1)(\alpha^2-1)\Big[\big(\alpha E(\alpha\lambda)\widehat{K}(-\alpha\lambda) - E(\lambda)\widehat{K}(-\lambda)\big) \times \big(E(\alpha\lambda) - E(\lambda)\big)\\
&\quad - \big(E(\alpha\lambda)\widehat{K}(-\alpha\lambda) - E(\lambda)\widehat{K}(-\lambda)\big) \times \big(\alpha E(\alpha\lambda) - E(\lambda)\big)\Big]\\
=&-3\alpha\Big[E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda)E(\lambda)(1-\alpha^2) + E(\lambda)\widehat{K}(-\lambda)E(\alpha^2\lambda)(\alpha^2-1)\Big]\\
&+3\Big[E(\alpha\lambda)\widehat{K}(-\alpha\lambda)E(\alpha^2\lambda)(\alpha^2-\alpha) + E(\alpha\lambda)\widehat{K}(-\alpha\lambda)E(\lambda)(\alpha-1) + E(\lambda)\widehat{K}(-\lambda)E(\alpha^2\lambda)(1-\alpha^2)\Big]\\
&+3\Big[E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda)E(\alpha\lambda)(\alpha-\alpha^2) + E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda)E(\lambda)(\alpha^2-1) + E(\lambda)\widehat{K}(-\lambda)E(\alpha\lambda)(1-\alpha)\Big]\\
&-3\alpha^2\Big[E(\alpha\lambda)\widehat{K}(-\alpha\lambda)E(\lambda)(1-\alpha) + E(\lambda)\widehat{K}(-\lambda)E(\alpha\lambda)(\alpha-1)\Big]\\
=&3\Big[E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda)E(\lambda)(\alpha^2-\alpha) - E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda)E(\alpha\lambda)(\alpha^2-\alpha)\\
&+E(\lambda)\widehat{K}(-\lambda)E(\alpha\lambda)(\alpha^2-\alpha) - E(\lambda)\widehat{K}(-\lambda)E(\alpha^2\lambda)(\alpha^2-\alpha)\\
&+E(\alpha\lambda)\widehat{K}(-\alpha\lambda)E(\alpha^2\lambda)(\alpha^2-\alpha) - E(\alpha\lambda)\widehat{K}(-\alpha\lambda)E(\lambda)(\alpha^2-\alpha)\Big]\\
=&3(\alpha^2-\alpha)\Big[E(\lambda)\widehat{K}(-\lambda)\big(E(\alpha\lambda) - E(\alpha^2\lambda)\big)\\
&\qquad + E(\alpha\lambda)\widehat{K}(-\alpha\lambda)\big(E(\alpha^2\lambda) - E(\lambda)\big)\\
&\qquad + E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda)\big(E(\lambda) - E(\alpha\lambda)\big)\Big]
\end{aligned}$$

We find the unknowns:

$$-i\lambda f_1(\lambda;1) = \frac{1}{\Delta(\lambda)}\Bigg[-3\alpha\Big[(Q(\alpha^2\lambda) - Q(\lambda)) \times (E(\alpha^2\lambda) - E(\lambda))$$
$$- \big(E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda) - E(\lambda)\widehat{K}(-\lambda)\big) \times (N(\alpha^2\lambda) - N(\lambda))\Big]$$
$$+3\Big[\big(E(\alpha\lambda)\widehat{K}(-\alpha\lambda) - E(\lambda)\widehat{K}(-\lambda)\big) \times \big(N(\alpha^2\lambda) - N(\lambda)\big)$$
$$- \big(Q(\alpha\lambda) - Q(\lambda)\big) \times \big(E(\alpha^2\lambda) - E(\lambda)\big)\Big]$$
$$+3\Big[\big(E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda) - E(\lambda)\widehat{K}(-\lambda)\big) \times \big(N(\alpha\lambda) - N(\lambda)\big)$$
$$- \big(Q(\alpha^2\lambda) - Q(\lambda)\big) \times \big(E(\alpha\lambda) - E(\lambda)\big)\Big]$$
$$-3\alpha^2\Big[\big(Q(\alpha\lambda) - Q(\lambda)\big) \times \big(E(\alpha\lambda) - E(\lambda)\big)$$
$$- \big(E(\alpha\lambda)\widehat{K}(-\alpha\lambda) - E(\lambda)\widehat{K}(-\lambda)\big) \times \big(N(\alpha\lambda) - N(\lambda)\big)\Big]\Bigg]$$

further simplifying we get

$$-i\lambda f_1(\lambda;1) = \frac{3}{\Delta(\lambda)}\Bigg[Q(\alpha^2\lambda)\Big[-\alpha E(\alpha^2\lambda) + \alpha E(\lambda) - E(\alpha\lambda) + E(\lambda)\Big]$$
$$+Q(\alpha\lambda)\Big[-E(\alpha^2\lambda) + E(\lambda) - \alpha^2 E(\alpha\lambda) + \alpha^2 E(\lambda)\Big]$$
$$+Q(\lambda)\Big[-\alpha E(\lambda) + E(\alpha^2\lambda) + \alpha E(\alpha^2\lambda) - E(\lambda) + \alpha^2 E(\alpha\lambda) - E(\lambda) + E(\alpha\lambda) - \alpha^2 E(\lambda)\Big]$$
$$+E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda)\Big[-N(\lambda) + \alpha N(\alpha^2\lambda) + N(\alpha\lambda) - \alpha N(\lambda)\Big]$$
$$+E(\alpha\lambda)\widehat{K}(-\alpha\lambda)\Big[-\alpha^2 N(\lambda) + N(\alpha^2\lambda) - N(\lambda) + \alpha^2 N(\alpha\lambda)\Big]$$
$$+E(\lambda)\widehat{K}(-\lambda)\Big[-\alpha N(\alpha^2\lambda) + \alpha N(\lambda) - N(\alpha^2\lambda) + N(\lambda) - N(\alpha\lambda) + N(\lambda) - \alpha^2 N(\alpha\lambda) + \alpha^2 N(\lambda)\Big]\Bigg]$$

$$
\begin{aligned}
-f_2(\lambda;1) = \frac{1}{\Delta(\lambda)}\Bigg[\quad & -3\alpha\Big[(\alpha^2 E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda) - E(\lambda)\widehat{K}(-\lambda)) \times (N(\alpha^2\lambda) - N(\lambda)) \\
& \qquad - \big(Q(\alpha^2\lambda) - Q(\lambda)\big) \times (\alpha^2 E(\alpha^2\lambda) - E(\lambda))\Big] \\
& +3\Big[\big(Q(\alpha\lambda) - Q(\lambda)\big) \times \big(\alpha^2 E(\alpha^2\lambda) - E(\lambda)\big) \\
& \qquad - \big(\alpha E(\alpha\lambda)\widehat{K}(-\alpha\lambda) - E(\lambda)\widehat{K}(-\lambda)\big) \times \big(N(\alpha^2\lambda) - N(\lambda)\big)\Big] \\
& +3\Big[\big(Q(\alpha^2\lambda) - Q(\lambda)\big) \times \big(\alpha E(\alpha\lambda) - E(\lambda)\big) \\
& \qquad - \big(\alpha^2 E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda) - E(\lambda)\widehat{K}(-\lambda)\big) \times \big(N(\alpha\lambda) - N(\lambda)\big)\Big] \\
& -3\alpha^2\Big[\big(\alpha E(\alpha\lambda)\widehat{K}(-\alpha\lambda) - E(\lambda)\widehat{K}(-\lambda)\big) \times \big(N(\alpha\lambda) - N(\lambda)\big) \\
& \qquad - \big(Q(\alpha\lambda) - Q(\lambda)\big) \times \big(\alpha E(\alpha\lambda) - E(\lambda)\big)\Big]\Bigg]
\end{aligned}
$$

further simplifying we get

$$
\begin{aligned}
-f_2(\lambda;1) =& \frac{3}{\Delta(\lambda)}\Bigg[Q(\alpha^2\lambda)\Big[E(\alpha^2\lambda) - \alpha E(\lambda) + \alpha E(\alpha\lambda) - E(\lambda)\Big] \\
& +Q(\alpha\lambda)\Big[\alpha^2 E(\alpha^2\lambda) - E(\lambda) + E(\alpha\lambda) - \alpha^2 E(\lambda)\Big] \\
& +Q(\lambda)\Big[\alpha E(\lambda) - E(\alpha^2\lambda) - \alpha^2 E(\alpha^2\lambda) + E(\lambda) - \alpha E(\alpha\lambda) + E(\lambda) - E(\alpha\lambda) + \alpha^2 E(\lambda)\Big] \\
& +E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda)\Big[N(\lambda) - N(\alpha^2\lambda) - \alpha^2 N(\alpha\lambda) + \alpha^2 N(\lambda)\Big] \\
& +E(\alpha\lambda)\widehat{K}(-\alpha\lambda)\Big[\alpha N(\lambda) - \alpha N(\alpha^2\lambda) + N(\lambda) - N(\alpha\lambda)\Big] \\
& +E(\lambda)\widehat{K}(-\lambda)\Big[\alpha N(\alpha^2\lambda) - \alpha N(\lambda) + N(\alpha^2\lambda) - N(\lambda) + N(\alpha\lambda) - N(\lambda) + \alpha^2 N(\alpha\lambda) - \alpha^2 N(\lambda)\Big]\Bigg]
\end{aligned}
$$

Hence, the sum $i\lambda f_1(\lambda;1)+f_2(\lambda;1)$ is equal to the following:

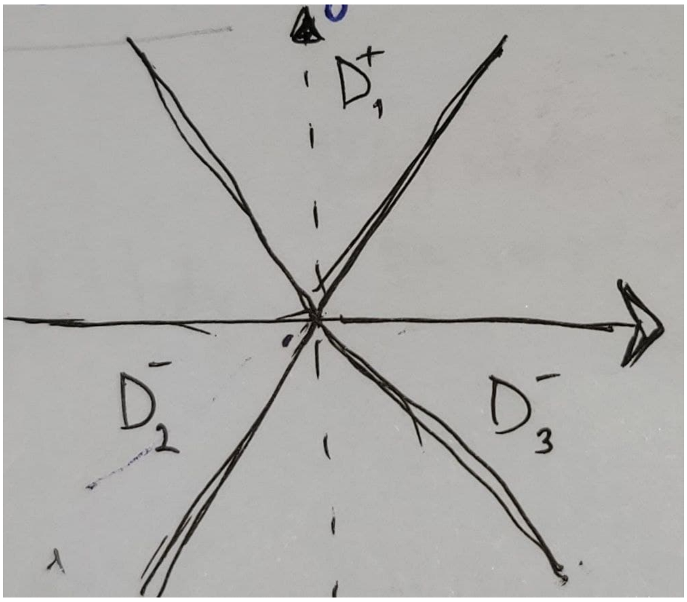
$$\begin{aligned}
i\lambda f_1(\lambda;1)+f_2(\lambda;1) = {} & \frac{-3}{\Delta(\lambda)}\Bigg[Q(\alpha^2\lambda)\Big[(1-\alpha)E(\alpha^2\lambda)-(1-\alpha)E(\alpha\lambda)\Big] \\
& +Q(\alpha\lambda)\Big[(\alpha^2-1)E(\alpha^2\lambda)-(\alpha^2-1)E(\alpha\lambda)\Big] \\
& +Q(\lambda)\Big[(\alpha-\alpha^2)E(\alpha^2\lambda)-(\alpha-\alpha^2)E(\alpha\lambda)\Big] \\
& +E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda)\Big[(\alpha-1)N(\alpha^2\lambda)+(1-\alpha^2)N(\alpha\lambda)+(\alpha^2-\alpha)N(\lambda)\Big] \\
& -E(\alpha\lambda)\widehat{K}(-\alpha\lambda)\Big[(\alpha-1)N(\alpha^2\lambda)+(1-\alpha^2)N(\alpha\lambda)+(\alpha^2-\alpha)N(\lambda)\Big]\Bigg] \\
= {} & \frac{-3}{\Delta(\lambda)}\Big[(\alpha-\alpha^2)\big(E(\alpha^2\lambda)-E(\alpha\lambda)\big)\big(\alpha^2Q(\alpha^2\lambda)+\alpha Q(\alpha\lambda)+Q(\lambda)\big) \\
& +(\alpha^2-\alpha)\big(E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda)-E(\alpha\lambda)\widehat{K}(-\alpha\lambda)\big)\big(\alpha^2N(\alpha^2\lambda)+\alpha N(\alpha\lambda)+N(\lambda)\big)\Big] \\
= {} & \frac{-3}{\Delta(\lambda)}(\alpha-\alpha^2)\Big[\big(E(\alpha^2\lambda)-E(\alpha\lambda)\big)\big(\alpha^2Q(\alpha^2\lambda)+\alpha Q(\alpha\lambda)+Q(\lambda)\big) \\
& -\big(E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda)-E(\alpha\lambda)\widehat{K}(-\alpha\lambda)\big)\big(\alpha^2N(\alpha^2\lambda)+\alpha N(\alpha\lambda)+N(\lambda)\big)\Big]
\end{aligned}$$

Now we can proceed to the asymptotic analysis within $cl(\widetilde{D}^-)$ since $i\lambda f_1(\lambda;1)+f_2(\lambda;1)$ is the sum inside the $\int_{\partial\widetilde{D}^-}$ as $\lambda\to\infty$ in the EFt (1). However, first we do the back substitution for $Q(\lambda)$ and $N(\lambda)$ only keeping the unknown terms in (4) and (5):

$$Q(\lambda)\approx\int_0^1 K(y)e^{i\lambda y}\widehat{q}(\lambda,t;y,1)dy \qquad N(\lambda)\approx\widehat{q}(\lambda,t;0,1)=\int_0^1 e^{-i\lambda x}q(x,t)dx$$

Now we need to show the decaying of the following ratio term inside $\int_{\partial\widetilde{D}^-}$ as $\lambda\to\infty$:

$$\begin{aligned}
& \Big[\big(E(\alpha^2\lambda)-E(\alpha\lambda)\big)\big(\alpha^2Q(\alpha^2\lambda)+\alpha Q(\alpha\lambda)+Q(\lambda)\big) \\
& -\big(E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda)-E(\alpha\lambda)\widehat{K}(-\alpha\lambda)\big)\big(\alpha^2N(\alpha^2\lambda)+\alpha N(\alpha\lambda)+N(\lambda)\big)\Big]\times \\
& \frac{1}{E(\lambda)\widehat{K}(-\lambda)\big(E(\alpha\lambda)-E(\alpha^2\lambda)\big)+E(\alpha\lambda)\widehat{K}(-\alpha\lambda)\big(E(\alpha^2\lambda)-E(\lambda)\big)+E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda)\big(E(\lambda)-E(\alpha\lambda)\big)}
\end{aligned} \tag{6}$$



4 Asymptotic analysis within $cl(\widetilde{D}_2^-)$

Decaying terms: $e^{-i\lambda}, e^{-i\alpha\lambda}, e^{i\alpha^2\lambda}$.
Blowing up terms: $e^{i\lambda}, e^{i\alpha\lambda}, e^{-i\alpha^2\lambda}$.

Note that we have already found the decay rate of all the terms inside the ratio term (6) within all three closures in Problem 1. Hence, within $cl(\widetilde{D}_2^-)$ the terms have the following decay rates as $\lambda \to \infty$:

$$\begin{aligned}
\widehat{K}(-\lambda) &= \int_0^1 K(y)e^{i\lambda y}dy = \mathcal{O}\left(\left|\frac{e^{i\lambda}}{\lambda}\right|\right)\\
\widehat{K}(-\alpha\lambda) &= \int_0^1 K(y)e^{i\alpha\lambda y}dy = \mathcal{O}\left(\left|\frac{e^{i\alpha\lambda}}{\lambda}\right|\right)\\
\widehat{K}(-\alpha^2\lambda) &= \int_0^1 K(y)e^{i\alpha^2\lambda y}dy = \mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)\\
N(\lambda) &\approx \widehat{q}(\lambda,t;0,1) = \int_0^1 e^{-i\lambda x}q(x,t)dx = \mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)\\
N(\alpha\lambda) &\approx \widehat{q}(\alpha\lambda,t;0,1) = \int_0^1 e^{-i\alpha\lambda x}q(x,t)dx = \mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)\\
N(\alpha^2\lambda) &\approx \widehat{q}(\alpha^2\lambda,t;0,1) = \int_0^1 e^{-i\alpha^2\lambda x}q(x,t)dx = \mathcal{O}\left(\left|\frac{e^{-i\alpha^2\lambda}}{\lambda}\right|\right)\\
Q(\lambda) &\approx \int_0^1 K(y)e^{i\lambda y}\widehat{q}(\lambda,t;y,1) = \mathcal{O}\left(\left|\frac{1}{\lambda^2}\right|\right)\\
Q(\alpha\lambda) &\approx \int_0^1 K(y)e^{i\alpha\lambda y}\widehat{q}(\alpha\lambda,t;y,1) = \mathcal{O}\left(\left|\frac{1}{\lambda^2}\right|\right)\\
Q(\alpha^2\lambda) &\approx \int_0^1 K(y)e^{i\alpha^2\lambda y}\widehat{q}(\alpha^2\lambda,t;y,1) = \mathcal{O}\left(\left|\frac{e^{-i\alpha^2\lambda}}{\lambda^2}\right|\right)
\end{aligned}$$

4.1 Denominator

Substituting above decay rates into the denominator in (6) we get

$$\begin{aligned}
&E(\lambda)\widehat{K}(-\lambda)\big(E(\alpha\lambda)-E(\alpha^2\lambda)\big)+E(\alpha\lambda)\widehat{K}(-\alpha\lambda)\big(E(\alpha^2\lambda)-E(\lambda)\big)+E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda)\big(E(\lambda)-E(\alpha\lambda)\big)=\\
=&\mathcal{O}\left(\left|e^{-i\lambda}\right|\right)\mathcal{O}\left(\left|\frac{e^{i\lambda}}{\lambda}\right|\right)\mathcal{O}\left(\left|e^{-i\alpha^2\lambda}\right|\right)+\mathcal{O}\left(\left|e^{-i\alpha\lambda}\right|\right)\mathcal{O}\left(\left|\frac{e^{i\alpha\lambda}}{\lambda}\right|\right)\mathcal{O}\left(\left|e^{-i\alpha^2\lambda}\right|\right)\\
&+\mathcal{O}\left(\left|e^{-i\alpha^2\lambda}\right|\right)\mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)\left(\mathcal{O}\left(\left|e^{-i\lambda}\right|\right)+\mathcal{O}\left(\left|e^{-i\alpha\lambda}\right|\right)\right)\\
=&\mathcal{O}\left(\left|\frac{e^{-i\alpha^2\lambda}}{\lambda}\right|\right)
\end{aligned}$$

4.2 Numerator

Note that the numerator of the ratio term (6) has some problematic blowing up terms that are multiplied:

$$\alpha^2E(\alpha^2\lambda)Q(\alpha^2\lambda)-\alpha^2E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda)N(\alpha^2\lambda)=-\alpha^2E(\alpha^2\lambda)\left(\widehat{K}(-\alpha^2\lambda)N(\alpha^2\lambda)-Q(\alpha^2\lambda)\right)$$

First, we examine $\widehat{K}(-\alpha^2\lambda)N(\alpha^2\lambda) - Q(\alpha^2\lambda)$:

$$
\begin{aligned}
&\widehat{q}(\alpha^2\lambda, t; 0, 1)\int_0^1 K(y)e^{i\alpha^2\lambda y}dy - \int_0^1 K(y)e^{i\alpha^2\lambda y}\widehat{q}(\alpha^2\lambda, t; y, 1)dy \\
=&\int_0^1 K(y)e^{i\alpha^2\lambda y}\widehat{q}(\alpha^2\lambda, t; 0, 1)dy - \int_0^1 K(y)e^{i\alpha^2\lambda y}\widehat{q}(\alpha^2\lambda, t; y, 1)dy \\
=&\int_0^1 K(y)e^{i\alpha^2\lambda y}\left(\widehat{q}(\alpha^2\lambda, t; 0, 1) - \widehat{q}(\alpha^2\lambda, t; y, 1)\right)dy \\
=&\int_0^1 e^{i\alpha^2\lambda y}K(y)\widehat{q}(\alpha^2\lambda, t; 0, y)dy \\
=&\frac{1}{i\alpha^2\lambda}\left[e^{i\alpha^2\lambda y}K(y)\widehat{q}(\alpha^2\lambda, t; 0, y)\right]_{y=0}^{y=1} - \frac{1}{i\alpha^2\lambda}\int_0^1 e^{i\alpha^2\lambda y}\left[K(y)e^{-i\alpha^2\lambda y}q(y,t) + K'(y)\widehat{q}(\alpha^2\lambda, t; 0, y)\right]dy \\
=&\frac{1}{i\alpha^2\lambda}e^{i\alpha^2\lambda}K(1)\widehat{q}(\alpha^2\lambda, t; 0, 1) - \frac{1}{i\alpha^2\lambda}\int_0^1 K(y)q(y,t)dy - \frac{1}{i\alpha^2\lambda}\int_0^1 e^{i\alpha^2\lambda y}K'(y)\widehat{q}(\alpha^2\lambda, t; 0, y)dy \\
=&\underbrace{\frac{1}{i\alpha^2\lambda}e^{i\alpha^2\lambda}K(1)\widehat{q}(\alpha^2\lambda, t; 0, 1)}_{=\mathcal{O}\left(\left|\frac{1}{\lambda^2}\right|\right)} - \underbrace{\frac{1}{i\alpha^2\lambda}g_0(t)}_{=\mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)} - \frac{1}{i\alpha^2\lambda}\int_0^1 e^{i\alpha^2\lambda y}K'(y)\widehat{q}(\alpha^2\lambda, t; 0, y)dy
\end{aligned}
$$

The problematic term in the equation above is $\frac{1}{i\alpha^2\lambda}g_0(t)$. However, since $g_0(t)$ is a known function given as a non-local condition, we can factor out this term outside of the integral.

Now rearranging the numerator in (6) and substituting the decay rates of individual terms we get

$$
\begin{aligned}
&\left(E(\alpha^2\lambda) - E(\alpha\lambda)\right)\left(\alpha^2 Q(\alpha^2\lambda) + \alpha Q(\alpha\lambda) + Q(\lambda)\right) \\
&- \left(E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda) - E(\alpha\lambda)\widehat{K}(-\alpha\lambda)\right)\left(\alpha^2 N(\alpha^2\lambda) + \alpha N(\alpha\lambda) + N(\lambda)\right) \\
=&E(\alpha^2\lambda)\left(\alpha Q(\alpha\lambda) + Q(\lambda)\right) - E(\alpha\lambda)\left(\alpha^2 Q(\alpha^2\lambda) + \alpha Q(\alpha\lambda) + Q(\lambda)\right) \\
&- E(\alpha^2\lambda)\widehat{K}(-\alpha^2\lambda)\left(\alpha N(\alpha\lambda) + N(\lambda)\right) + E(\alpha\lambda)\widehat{K}(-\alpha\lambda)\left(\alpha^2 N(\alpha^2\lambda) + \alpha N(\alpha\lambda) + N(\lambda)\right) \\
&- \alpha^2 E(\alpha^2\lambda)\left(\widehat{K}(-\alpha^2\lambda)N(\alpha^2\lambda) - Q(\alpha^2\lambda)\right) \\
=&\mathcal{O}\left(\left|e^{-i\alpha^2\lambda}\right|\right)\mathcal{O}\left(\left|\frac{1}{\lambda^2}\right|\right) - \mathcal{O}\left(\left|e^{-i\alpha\lambda}\right|\right)\mathcal{O}\left(\left|\frac{e^{-i\alpha^2\lambda}}{\lambda^2}\right|\right) - \mathcal{O}\left(\left|\frac{e^{-i\alpha^2\lambda}}{\lambda}\right|\right)\mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right) + \mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)\mathcal{O}\left(\left|\frac{e^{-i\alpha^2\lambda}}{\lambda}\right|\right) \\
&- \mathcal{O}\left(\left|e^{-i\alpha^2\lambda}\right|\right)\mathcal{O}\left(\left|\frac{1}{\lambda^2}\right|\right) \\
=&\mathcal{O}\left(\left|\frac{e^{-i\alpha^2\lambda}}{\lambda^2}\right|\right)
\end{aligned}
$$

Thus, the decay rate of the ratio term inside $\int_{\partial\widetilde{D_2^-}}$ is

$$
\frac{\mathcal{O}\left(\left|\frac{e^{-i\alpha^2\lambda}}{\lambda^2}\right|\right)}{\mathcal{O}\left(\left|\frac{e^{-i\alpha^2\lambda}}{\lambda}\right|\right)} = \mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)
$$

Hence, as $\lambda \to \infty$, this term approaches 0 within $cl(\widetilde{D}_2^-)$.

5 Asymptotic analysis within $cl(\widetilde{D}_3^-)$

Decaying terms: $e^{-i\lambda}, e^{i\alpha\lambda}, e^{-i\alpha^2\lambda}$.
Blowing up terms: $e^{i\lambda}, e^{-i\alpha\lambda}, e^{i\alpha^2\lambda}$.

Note that the decay rate of the ratio term (**??**), which can is a function of λ is independent of a particular λ mapping. Moreover, by applying a particular mapping $\lambda \to \alpha\lambda$ to the decaying and blowing up terms within $cl(\widetilde{D}_3^-)$ we can retrieve the decaying and the blowing up terms within $cl(\widetilde{D}_2^-)$. These two facts are enough to conclude that the decay rate of the entire ratio term (**??**) within $cl(\widetilde{D}_3^-)$ is the same as the decay rate within $cl(\widetilde{D}_2^-)$. So, the decay rate of the ratio term inside $\int_{\partial \widetilde{D}_3^-}$ is $\mathcal{O}\left(\left|\frac{1}{\lambda}\right|\right)$ Thus, as $\lambda \to \infty$, this term approaches 0 within $cl(\widetilde{D}_3^-)$.