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TIME-PERIODIC NON-LOCAL PROBLEM FOR THE STOKES EQUATION

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Abstract

B.Sc (Hons)

TIME-PERIODIC NON-LOCAL PROBLEM FOR THE STOKES EQUATION

by Bekzod NORMATOV

We study the long time behaviour of the solution of the Stokes equation on a finite interval when the prescribed boundary and non-local conditions are time-periodic. We adapt the *Q-equation* approach pioneered by A. S. Fokas and M. C. Van der Weele in 2021 to solve the problem in a time-periodic regime and then reconstruct the solution to the problem with an explicit initial condition. We show that the transient effect of the initial condition may be ignored in the long time limit under certain assumptions. Finally, we formulate a useful conjecture encompassing the main results found in this project.

Contents

Chapter 1

Introduction

1.1 Motivation

An Initial Boundary Value Problem (IBVP) is a type of mathematical problem that involves finding the solution to a differential equation, given certain initial conditions and boundary conditions (Evans, 2022). The initial conditions specify the values of the solution and its derivatives at a particular initial time, while the boundary conditions specify the behavior of the solution at the boundaries of the domain over which it is defined. For example, the heat equation is a common initial boundary value problem, where the temperature at different points in a material is modeled over time, subject to initial and boundary conditions (Reed, Simon, et al., 1980). Another example is the wave equation, which describes the behavior of waves, such as sound or light, subject to similar conditions.

Initial Boundary Value Problems come in various shapes and sizes and solving them can be a challenging task, but it is a fundamental problem in many areas of science and engineering, from fluid mechanics to electromagnetics. In this project, we consider IBVPs posed on two-dimensional domains of independent variables, position x and time t, with corresponding initial condition, boundary conditions, and a partial differential equation describing a physical system.

There are many ways to set up these problems making them more difficult or easier to solve. The classical method involving the separation of variables and the use of the Fourier series to solve IBVPs was proposed by Joseph Fourier in 1822 and has been extensively studied (Fourier, 1822). However, this method falls short in addressing IBVPs that involve at least one of the following:

- IBVP with inhomogeneous boundary conditions, e.g. $u_x(0,t) = g(t)$, where $g(t)$ is a known function of time (unless $g(t)$ decays in t).
- IBVP with non-local conditions, which describe a weighted average of the solution, e.g. $\int_0^1 K(x)u(x,t)dx = h(t)$, where $K(x), h(t)$ are both known functions.
- IBVP with high order PDEs, e.g. the Stokes equation $u_t = -u_{xxx}$.

A relatively new method for solving this class of more general IBVPs was developed in 1997 by Fokas, and hence named the Fokas Transform Method, also known as the Unified Transform Method (Fokas, 1997). A thorough introduction to the method can be found in (Fokas, 2008) and (Deconinck, Trogdon, and Vasan, 2014). The method can be customized to solve a whole class of IBVPs with higher-order PDEs, non-local conditions, and inhomogeneous boundary conditions. The method utilizes the Fourier Transform and allows for an explicit contour integral solution representation that cannot be obtained using a classical method. The ongoing research on the Fokas method has advanced our understanding of higher-order PDEs greatly. However, the Unified Transform Method requires significant mathematical background and expertise to use effectively, and the derivation of the solution can be time-consuming. That is where lies the benefit of the most novel approach to solving IBVPs based on the so-called *Q-equation*. The method was pioneered by Van der Weele and Fokas in 2021 (Fokas and Weele, 2021).

1.2 Setting up the Problem

The project considers a specific Initial Non-local Boundary Value Problem with the Stokes equation, inhomogeneous time-periodic boundary and non-local conditions, and an explicit initial condition on a finite interval L. To reduce the amount of notation used, we assume, without loss of generality, that L has a value of 1. We can do that due to the rescaling argument (Davenport, 2017). In particular, we consider a problem with one non-local (1.1.NC), two boundary conditions (1.1.BC1) and (1.1.BC2) and an explicit initial condition (1.1.IC). The Problem is set up below.

$$u_t(x,t) + u_{xxx}(x,t) = 0 \qquad (x,t) \in [0,1] \times [0,\infty), \tag{1.1.PDE}$$

$$u(x,0) = U(x) \qquad x \in [0,1], \tag{1.1.IC}$$

$$u(1,t) = h_0(t) \qquad t \in [0,\infty), \tag{1.1.BC1}$$

$$u_x(1,t) = h_1(t) \qquad t \in [0,\infty), \tag{1.1.BC2}$$

$$\int_0^1 K(y)u(y,t)dy = a(t) \qquad t \in [0,\infty), \tag{1.1.NC}$$

where functions $U(x)$, $h_0(t)$, $h_1(t)$, $a(t)$ and $K(y)$ are all known and well-defined. We will come back to this problem in the later chapter, and hereon it will be referred to as **Problem** (1.1). By adapting the *Q-equation* approach to this problem, we aim to:

1. Show that for large t the solution is indeed asymptotically time-periodic.

2. Find an explicit, asymptotically valid time-periodic representation for the solution.

1.3 About *Q-equation* formalism

The *Q-equation* formalism relies on building Dirichlet-to-Neumann, also referred to as Data-to-Unknown (D-to-N) mapping. D-to-N mapping provides a simple, algebraic way of finding unknown boundary values from the known data of the problem. The *Q-equation* relates Fourier coefficients of the boundary values and a Fourier transform of the solution of the problem. Therefore, by reconstructing the boundary values from the Fourier coefficients it provides a map from the data to the unknown boundary values, a D-to-N map.

1.3.1 Time-periodicity assumption

The *Q-equation* method necessitates the assumption of the time-periodicity of the solution. However, this assumption, in turn, introduces an error term, which comes from the difference in the initial conditions of two problem setups: the implicit initial condition that ensures a time-periodic solution might differ from the explicit initial condition in the IBVP we actually wish to study. The given IBVP may not truly be time-periodic but tend towards a time-periodic solution for large t. Hence, a necessary step in any application of the *Q-equation* method is the asymptotic analysis of the behavior of the error term for large t. In other words, one requirement is to show that for large t the solution without any assumption of time-periodicity tends towards a time-periodic solution.

First, we need to extend the *Q-equation* method to problems with non-local data to express a non-local *Q-equation*. Then, using the non-local *Q-equation* we can find the unknown boundary values. Once all the boundary values with a common period T are found, and the analyticity conditions hold, we can create an explicit initial condition that ensures the solution to the corresponding problem is exactly T-periodic. Having found the time-periodic solution allows us to establish the error term and asymptotically analyze it for large t. As mentioned earlier, we expect to show that the error term decays in time.

1.4 Why we study the Stokes equation

The Stokes equation (1.3) is a useful partial differential equation from both an applied mathematics and a pure mathematics point of view. It is a simplification of the Korteweg–De Vries (KdV) equation (1.2) for u very small, and the main reason why we study the Stokes equation instead of the KdV is that the Stokes equation is mathematically more tractable given the scope of this paper. Both equations can be seen below.

$$u_t + u_{xxx} + uu_x = 0 \tag{1.2}$$

$$u_t + u_{xxx} = 0 \tag{1.3}$$

The Korteweg–De Vries equation (1.2) was originally derived to describe the behavior of long waves traveling in shallow canals. This equation is a nonlinear partial differential equation that describes the evolution of a certain type of waves known as solitons (Korteweg and De Vries, 1895). The KdV equation can also have applications in various other physical systems such as tsunamis in the open ocean (Yaacob, Sarif, and Aziz, 2008). That is because the main assumption made in the derivation of the KdV equation is that the wavelength of the wave is much longer than the depth of the water (Korteweg and De Vries, 1895). This assumption is known as the shallow water assumption, and it is satisfied in the open ocean, where the wavelength of a tsunami is typically on the order of hundreds of kilometers or more, while the depth of the ocean is typically on the order of a few kilometers (IOC-UNESCO, 2019).

Another important assumption made in both the KdV and the Stokes equation is that displacement in the second spatial dimension y is ignored. This simplification is made because there are situations where variation in the second spatial dimension is extremely small and hence negligible. For example, after a tsunami has traveled some distance its wavefront has little variation in the alongshore direction while having lots of variation in the perpendicular to the shore direction. These are the situations that are interesting to us in this paper.

Overall, the two assumptions explained above are both simplifying and relevant as they are realistic in certain physical systems. These systems are the ones we want to examine and model using the Stokes equation (1.3).

1.5 Why we study D-to-N maps

There are two main reasons why Data-to-Unknown mapping is useful in solving non-local initial boundary value problems. First is that in some physical systems that can be described using the Stokes equation or another PDE, we might be primarily interested in what happens at the boundaries of the system. For example, consider a tsunami wave in the open ocean. The most interesting or even important behavior of the tsunami would be where it touches the ground, the bridge, or the wall, which in other words is the boundary of the open ocean. In cases like this where we are interested in the behavior of the system at its boundary solving directly for the boundary conditions is more sensible and useful than finding a general solution first.

We might also solve for the unknown boundary values because in some situations solving D-to-N maps makes it easier to solve the full problem. Indeed, having equations for all boundary values allows us to reconstruct the solution as can be studied in (Deconinck, Trogdon, and Vasan, 2014). Hence, we rely on the *Q-equation formalism* that uses D-to-N mapping to solve **Problem** (1.1).

Chapter 2

Solving the Problem

2.1 Methodology

We use the same method as in (Fokas, Pelloni, and Smith, 2022) of finding unknown boundary values in problems with time-periodic boundary conditions. However, instead of a problem with 3 time-periodic boundary conditions, we adapt the method to a problem with both boundary and non-local conditions as well as a problem with only non-local conditions. Non-local conditions allow for generality that is lost when we consider explicit boundary conditions. The methodology of finding the unknown time-periodic boundary values relies on the definition of the *Q-equation* and the solutions rest on the analysis of this equation. Following the same method as in (Miller and Smith, 2018) and (Pelloni and Smith, 2018) for solving problems with non-local conditions, we first obtain a non-local formulation of the *Q-equation*.

As mentioned earlier, non-local conditions can be seen as generalizations of boundary conditions as they capture the behavior of the solution of the problem over some finite interval. Hence, we first adapt the methodology used in (Fokas, Pelloni, and Smith, 2022) to the following Initial Non-local Boundary Value Problem with the Stokes equation, an explicit initial condition and three time-periodic non-local conditions, with common period T:

$$\begin{aligned}
u_t(x,t) + u_{xxx}(x,t) &= 0 && (x,t) \in [0,1] \times [0,\infty), \\
\int_0^1 K_0(y)u(y,t)dy &= a(t) && t \in [0,\infty), \\
\int_0^1 K_1(y)u(y,t)dy &= b(t) && t \in [0,\infty), \\
\int_0^1 K_2(y)u(y,t)dy &= c(t) && t \in [0,\infty), \\
u(x,0) &=: U(x) && x \in [0,1].
\end{aligned} \tag{2.1}$$

Problem (2.1) is posed for $x \in [0,1]$ and $t \geq 0$ and functions $a(t)$, $b(t)$, $c(t)$, $K_0(y)$, $K_1(y)$, $K_2(y)$ and $U(x)$ are all known and well-defined. The first building block of finding the unknown time-periodic boundary terms is to set up a new problem in a time-periodic regime with a period T solution and an implicit initial condition that depends on the time-periodic solution. This problem is set up below:

$$\begin{aligned}
q_t(x,t) + q_{xxx}(x,t) &= 0 && (x,t) \in [0,1] \times [0,\infty), \\
\int_0^1 K_0(y)q(y,t)dy &= a(t) && t \in [0,\infty), \\
\int_0^1 K_1(y)q(y,t)dy &= b(t) && t \in [0,\infty), \\
\int_0^1 K_2(y)q(y,t)dy &= c(t) && t \in [0,\infty), \\
q(x,t) &= q(x,t+T) && (x,t) \in [0,1] \times [0,\infty).
\end{aligned} \tag{2.2}$$

Problem (2.2) is also posed for $x \in [0,1]$ and $t \geq 0$ and functions $a(t)$,

$b(t), c(t), K_0(y), K_1(y), K_2(y)$ are all known and well-defined. Moreover, the time-periodicity assumption is represented by $q(x,t) = q(x,t+T)$, where $T \geq 0$ is a common period. Having set up these two problems we can move to the general method used to solve Problem (2.1).

The Method:

1. Calculate the non-local *Q-equation* using a similar method to (Miller and Smith, 2018). This will relate Fourier coefficients of the non-local values a, b, c with the Fourier coefficients of the boundary values on one side, e.g. at $x = 1$.

2. Calculate the period T Fourier coefficients A_j, B_j, C_j for $j \in \mathbb{Z}$ of the non-local values a, b, c.

3. Solve the non-local *Q-equation* for $H_j^{(0)}, H_j^{(1)}, H_j^{(2)}$, which are the Fourier coefficients of period T Fourier expansions of all the boundary terms on one side, e.g. at $x = 1$. *Note:* this might not work, the matrix might be singular!

4. Calculate the usual *Q-equation* with 6 boundary values.

5. We already have found the Fourier coefficients of the period T Fourier expansions of all the boundary values on one side, i.e. $H_j^{(0)}, H_j^{(1)}, H_j^{(2)}$. Hence, by solving the usual *Q-equation*, calculate the period T Fourier coefficients $G_j^{(0)}, G_j^{(1)}, G_j^{(2)}$ of the boundary values on the other side, e.g. at $x = 0$. *Note:* this might not work, the matrix might be singular!

6. Use the usual *Q-equation* to obtain $q_j(\lambda)$ for $j \in \mathbb{Z}$, which are the period T Fourier coefficients of the temporal Fourier expansion of $\widehat{q}(\lambda, t)$, where the latter is a spatial Fourier transform of $q(x,t)$.

7. Use Fourier inversion to obtain $q(x,t)$. This solves Problem (2.2).

8. Evaluate $q(x,0) =: Q(x)$. If $Q(x) = U(x)$, then Problem (2.1) and Problem (2.2) have the same solution. However, usually, that is not the case.

9. Set up a new problem with $v(x,t)$ as the unknown such that $v(x,t) = u(x,t) - q(x,t)$:

$$
\begin{aligned}
v_t(x,t) + v_{xxx}(x,t) &= 0 && (x,t) \in [0,1] \times [0,\infty), \\
\int_0^1 K_0(y)v(y,t)dy &= 0 && t \in [0,\infty), \\
\int_0^1 K_1(y)v(y,t)dy &= 0 && t \in [0,\infty), \\
\int_0^1 K_2(y)v(y,t)dy &= 0 && t \in [0,\infty), \\
v(x,0) &=: V(x) && x \in [0,1].
\end{aligned}
$$

Note that all three non-local conditions are now homogeneous.

10. Solve for v. The solution has already been found in the research project conducted by the author of this paper and the supervisor (Smith and Normatov, Accessed 2023-03-24). Hence, reusing that work we have an explicit contour integral solution for $v(x,t)$.

11. $u(x,t) = v(x,t) + q(x,t)$ solves Problem (2.1). However, on top of solving for u, we want to show that the solution is asymptotically time-periodic for large t. That is equivalent to saying that for large t, $v(x,t)$ approaches zero. Moreover, we want to analyze how fast function v decays in time. In order to do that we conduct an asymptotic analysis of $v(x,t)$ and we formulate a conjecture.

Remark: Note that while the problem setup used in the methodology above has no boundary conditions, the method can still be applied and will be applied to problems where some non-local conditions are replaced by boundary conditions. In fact, boundary conditions make the problem algebraically simpler as the dimension of the *Q-equation* matrix is likely to decrease.

If we succeed in proving the conjecture mentioned in step 11, which will be formulated later, we will be able to show that for given time-periodic non-local and boundary data, the solution $u(x,t)$ is asymptotically time-periodic. Generalizing this to the necessary conditions for asymptotic periodicity is outside the scope of this paper. However, the analysis of the solution to the problem considered in this paper provides useful insight into what conditions are necessary.

2.2 Problem setup

Now, we apply the above methodology to a specific problem that we consider in this paper. Recall **Problem** (1.1), an Initial Non-local Boundary Value Problem with inhomogeneous time-periodic boundary and non-local conditions and an explicit initial condition.

In order to solve **Problem** (1.1), we set up a new problem, **Problem** (2.3), in a time-periodic regime. This problem has the same inhomogeneous time-periodic boundary and non-local conditions. However, **Problem** (2.3) does not have a prescribed explicit initial condition compared to **Problem** (1.1). Instead, it has a periodicity condition represented by equation (2.3.PC), where $T \geq 0$ is a common period:

$$
\begin{aligned}
q_t(x,t) + q_{xxx}(x,t) &= 0 && (x,t) \in [0,1] \times [0,\infty), && \text{(2.3.PDE)}\\
q(x,t) &= q(x,t+T) && (x,t) \in [0,1] \times [0,\infty), && \text{(2.3.PC)}\\
q(1,t) &= h_0(t) && t \in [0,\infty), && \text{(2.3.BC1)}\\
q_x(1,t) &= h_1(t) && t \in [0,\infty), && \text{(2.3.BC2)}\\
\int_0^1 K(y)q(y,t)dy &= a(t) && t \in [0,\infty). && \text{(2.3.NC)}
\end{aligned}
$$

First, we aim to find the unknown boundary values in **Problem** (2.3). We use the same method as outlined in the previous section.

1. We use $y, z \in [0,1]$ with $y \leq z$, which are variable artificial boundaries of x. We already know the usual *Q-equation* for these boundaries:

$$
\begin{aligned}
\left[\frac{\partial}{\partial t} - i\lambda^3\right]\widehat{q}(\lambda;t,y,z) &= e^{-i\lambda y}\left(q_{xx}(y,t) + i\lambda q_x(y,t) - \lambda^2 q(y,t)\right)\\
&\quad - e^{-i\lambda z}\left(q_{xx}(z,t) + i\lambda q_x(z,t) - \lambda^2 q(z,t)\right).
\end{aligned}
\tag{2.4}
$$

   We evaluate the above equation at $z = 1$ and multiply by $e^{i\lambda y}K(y)$:

$$
\begin{aligned}
&\left[\frac{\partial}{\partial t} - i\lambda^3\right] e^{i\lambda y}K(y)\widehat{q}(\lambda;t,y,1)\\
&\qquad = K(y)\Big(q_{xx}(y,t) + i\lambda q_x(y,t) - \lambda^2 q(y,t)\Big)\\
&\qquad\quad - e^{-i\lambda(1-y)}K(y)\Big(q_{xx}(1,t) + i\lambda q_x(1,t) - \lambda^2 q(1,t)\Big).
\end{aligned}
$$

Now we integrate in y from 0 to 1:

$$\begin{aligned}\left[\frac{\partial}{\partial t} - i\lambda^3\right] \int_0^1 & e^{i\lambda y} K(y)\widehat{q}(\lambda; t, y, 1)dy \\ &= \int_0^1 K(y)q_{xx}(y,t)dy + i\lambda \int_0^1 K(y)q_x(y,t)dy \\ &\quad - \lambda^2 \int_0^1 K(y)q(y,t)dy \\ &\quad - \int_0^1 e^{-i\lambda(1-y)}K(y)\Big(q_{xx}(1,t) + i\lambda q_x(1,t) - \lambda^2 q(1,t)\Big)dy.\end{aligned}$$

The equation above is the non-local *Q-equation* as it contains both non-local and boundary terms. Now we can substitute the non-local and boundary data into the non-local *Q-equation*:

$$\begin{aligned}\left[\frac{\partial}{\partial t} - i\lambda^3\right] \int_0^1 & e^{i\lambda y} K(y)\widehat{q}(\lambda; t, y, 1)dy \\ &= \int_0^1 K(y)q_{xx}(y,t)dy + i\lambda \int_0^1 K(y)q_x(y,t)dy \\ &\quad - \lambda^2 a(t) - \widehat{K}(\lambda)\Big(q_{xx}(1,t) + i\lambda h_1(t) - \lambda^2 h_0(t)\Big) \\ &= \int_0^1 K(y)q_{xx}(y,t)dy + i\lambda \int_0^1 K(y)q_x(y,t)dy \\ &\quad - \widehat{K}(\lambda)q_{xx}(1,t) - \lambda^2 a(t) - i\lambda\widehat{K}(\lambda)h_1(t) + \lambda^2\widehat{K}(\lambda)h_0(t)\end{aligned} \tag{2.5}$$

for $\widehat{K}(\lambda) := \int_0^1 e^{-i\lambda(1-y)}K(y)dy$.

2. Before we calculate the period T Fourier coefficients, we denote a few functions. Let

$$
\begin{aligned}
h_2(t) &:= q_{xx}(1,t), \\
b(t) &:= \int_0^1 K(y) q_x(y,t) dy, \\
c(t) &:= \int_0^1 K(y) q_{xx}(y,t) dy.
\end{aligned}
$$

Moreover, for $\omega = \frac{2\pi}{T}$, denote

$$
\begin{aligned}
A_j &:= F_{ser}[a](j) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} a(t) e^{-ij\omega t} dt, \\
B_j &:= F_{ser}[b](j), \\
C_j &:= F_{ser}[c](j), \\
H_j^{(0)} &:= F_{ser}[h_0](j), \\
H_j^{(1)} &:= F_{ser}[h_1](j), \\
H_j^{(2)} &:= F_{ser}[h_2](j), \\
Q_j(\lambda) &:= F_{ser}\Big[\int_0^1 e^{i\lambda y} K(y) \widehat{q}(\lambda; \cdot, y, 1) dy \Big](j).
\end{aligned}
$$

Then, by the Fourier series representation for a periodic function in equation 4.11 from (Chaparro and Akan, 2018),

$$
\begin{aligned}
a(t) &= \sum_{j\in\mathbb{Z}} A_j e^{ij\omega t}, \\
b(t) &= \sum_{j\in\mathbb{Z}} B_j e^{ij\omega t}, \\
c(t) &= \sum_{j\in\mathbb{Z}} C_j e^{ij\omega t}, \\
h_0(t) &= \sum_{j\in\mathbb{Z}} H_j^{(0)} e^{ij\omega t}, \\
h_1(t) &= \sum_{j\in\mathbb{Z}} H_j^{(1)} e^{ij\omega t}, \\
h_2(t) &= \sum_{j\in\mathbb{Z}} H_j^{(2)} e^{ij\omega t}, \\
\int_0^1 e^{i\lambda y} K(y)\widehat{q}(\lambda; t, y, 1)dy &= \sum_{j\in\mathbb{Z}} Q_j(\lambda) e^{ij\omega t}, \\
\Rightarrow \frac{\partial}{\partial t}\int_0^1 e^{i\lambda y} K(y)\widehat{q}(\lambda; t, y, 1)dy &= \sum_{j\in\mathbb{Z}} (ij\omega) Q_j(\lambda) e^{ij\omega t}.
\end{aligned}
$$

3. All the equations above are the period T Fourier expansions of the corresponding boundary or non-local terms. Now we substitute these into the non-local *Q-equation* (2.5):

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}} e^{ij\omega t}(ij\omega - i\lambda^3)Q_j(\lambda) = \sum_{j\in\mathbb{Z}} e^{ij\omega t}\Big(C_j + i\lambda B_j - \widehat{K}(\lambda)H_j^{(2)} \\
- \lambda^2 A_j - i\lambda\widehat{K}(\lambda)H_j^{(1)} + \lambda^2\widehat{K}(\lambda)H_j^{(0)}\Big).
\end{aligned}
$$

We use the following corollary that is proven in Appendix 11.

Corollary 1. *If* $\forall x \in [-b, b]$,

$$\sum_{j\in\mathbb{Z}} e^{ij\pi x/b}\alpha_j = \sum_{j\in\mathbb{Z}} e^{ij\pi x/b}\beta_j.$$

Then, $\forall j \in \mathbb{Z}, \alpha_j = \beta_j$.

By the above corollary, $\forall j \in \mathbb{Z}$

$$\begin{aligned}(ij\omega - i\lambda^3)Q_j(\lambda) = C_j + i\lambda B_j - \widehat{K}(\lambda)H_j^{(2)} \\ - \lambda^2 A_j - i\lambda\widehat{K}(\lambda)H_j^{(1)} + \lambda^2\widehat{K}(\lambda)H_j^{(0)}.\end{aligned} \tag{2.6}$$

We want to argue that the left side of the above equation, hence also its right side, are zero for all λ such that $(ij\omega - i\lambda^3) = 0$. However, we should first check that $Q_j(\lambda)$ is finite for any such λ.

Lemma 2. *Suppose* $Q_j(\lambda)$ *is a function of Fourier coefficients defined by*

$$Q_j(\lambda) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\int_0^1 e^{i\lambda y}K(y)\widehat{q}(\lambda; t, y, 1)dy\, e^{-ij\omega t}dt, \quad \forall\lambda \in \mathbb{C}, \forall j \in \mathbb{Z}$$

where

$$\widehat{q}(\lambda; t, y, 1) = \int_y^1 e^{-i\lambda x}q(x, t)dx,$$

T *is a period of* $q(x, t)$*,* $\omega = \frac{2\pi}{T}$ *and* $K(y)$ *is a known function for* $y \in [0, 1]$*. Then,* $\forall\lambda \in \mathbb{C}, \forall j \in \mathbb{Z}$

$$|Q_j(\lambda)| < \infty.$$

The proof of the above lemma is provided in Appendex 12. Note that by Lemma 2, $|Q_j(\lambda)|$ is finite and the equation (2.6) holds $\forall\lambda \in$

$\mathbb{C}$. In particular, it holds for those λ for which the left-hand side of equation (2.6) is zero, i.e. those λ for which $ij\omega - i\lambda^3 = 0$. This simplifies equation (2.6) to

$$C_j + i\lambda B_j - \widehat{K}(\lambda)H_j^{(2)} = \lambda^2 A_j + i\lambda\widehat{K}(\lambda)H_j^{(1)} - \lambda^2\widehat{K}(\lambda)H_j^{(0)}. \quad (2.7)$$

For each $j \in \mathbb{Z} \setminus \{0\}$, there are three such λ: $\lambda_j, \alpha\lambda_j, \alpha^2\lambda_j$, where $\lambda_j = \sqrt[3]{j\omega}, \alpha = e^{\frac{2\pi}{3}i}$. Denote

$$N_j(\lambda) := \lambda^2 A_j + i\lambda\widehat{K}(\lambda)H_j^{(1)} - \lambda^2\widehat{K}(\lambda)H_j^{(0)}$$

as our known data. Now applying maps $\lambda \to \lambda_j, \lambda \to \alpha\lambda_j, \lambda \to \alpha^2\lambda_j$ to equation (2.7) forms a system of three linear equations with three unknowns:

$$\begin{pmatrix} 1 & i\lambda_j & -\widehat{K}(\lambda_j) \\ 1 & i\alpha\lambda_j & -\widehat{K}(\alpha\lambda_j) \\ 1 & i\alpha^2\lambda_j & -\widehat{K}(\alpha^2\lambda_j) \end{pmatrix} \begin{pmatrix} C_j \\ B_j \\ H_j^{(2)} \end{pmatrix} = \begin{pmatrix} N_j(\lambda_j) \\ N_j(\alpha\lambda_j) \\ N_j(\alpha^2\lambda_j) \end{pmatrix}.$$

We seek period T Fourier expansions of all the necessary boundary values on the right, i.e. $H_j^{(0)}, H_j^{(1)}, H_j^{(2)}$. Since, we already know $H_j^{(0)}, H_j^{(1)}$, we only need to find an equation for $H_j^{(2)}$ using the above system (2.8).

Remark: This might not work if the system is singular at $\lambda_j, \alpha\lambda_j, \alpha^2\lambda_j$ as defined above. To check that, one could find zeros of the matrix using a numerical root-finding algorithm based on the principal argument. If the system is singular, we cannot proceed further, and the result is that the given initial non-local boundary value problem does not have a time-periodic or asymptotically time-periodic solution!

We proceed under the assumption that system (2.8) is not singular. Then, the determinant is

$$\Delta(\lambda_j) = i\lambda_j(\alpha - \alpha^2)\left[\widehat{K}(\lambda_j) + \alpha\widehat{K}(\alpha\lambda_j) + \alpha^2\widehat{K}(\alpha^2\lambda_j)\right].$$

Now, we seek $H_j^{(2)}$ using Cramer's rule. The modified Cramer's matrix is

$$\begin{pmatrix} 1 & i\lambda_j & N_j(\lambda_j) \\ 0 & i(\alpha-1)\lambda_j & N_j(\alpha\lambda_j) - N_j(\lambda_j) \\ 0 & i(\alpha^2-1)\lambda_j & N_j(\alpha^2\lambda_j) - N_j(\lambda_j) \end{pmatrix},$$

and its determinant is equal to

$$\Delta_{H_j^{(2)}}(\lambda_j) = -i\lambda_j(\alpha - \alpha^2)\left[N_j(\lambda_j) + \alpha N_j(\alpha\lambda_j) + \alpha^2 N_j(\alpha^2\lambda_j)\right].$$

Then,

$$\begin{aligned} H_j^{(2)} &= \frac{\Delta_{H_j^{(2)}}(\lambda_j)}{\Delta(\lambda_j)} \\ &= -\frac{N_j(\lambda_j) + \alpha N_j(\alpha\lambda_j) + \alpha^2 N_j(\alpha^2\lambda_j)}{\widehat{K}(\lambda_j) + \alpha\widehat{K}(\alpha\lambda_j) + \alpha^2\widehat{K}(\alpha^2\lambda_j)}, \qquad \forall j \in \mathbb{Z}\setminus\{0\}. \end{aligned}$$

Now we seek $H_j^{(2)}$ for $j = 0$. Letting $j = 0$, equation (2.6) becomes

$$\begin{aligned} -i\lambda^3 Q_0(\lambda) = &\left(C_0 + i\lambda B_0 - \widehat{K}(\lambda)H_0^{(2)}\right) \\ &+ \left(-\lambda^2 A_0 - i\lambda\widehat{K}(\lambda)H_0^{(1)} + \lambda^2\widehat{K}(\lambda)H_0^{(0)}\right). \end{aligned}$$

Note that we are only interested in finding $H_0^{(2)}$. Letting $\lambda = 0$, we get

$$H_0^{(2)} = \frac{C_0}{\widehat{K}(0)}.$$

Finally, we have found $H_j^{(2)}$ $\forall j \in \mathbb{Z}$. Now, we can reconstruct the last boundary term on the right, i.e.

$$q_{xx}(1,t) = h_2(t) = \sum_{j \in \mathbb{Z}} H_j^{(2)} e^{ij\omega t}.$$

4. We evaluate the usual *Q-equation* (2.4) at $y = 0, z = 1$:

$$\left[\frac{\partial}{\partial t} - i\lambda^3\right] \widehat{q}(\lambda;t) = \big(q_{xx}(0,t) + i\lambda q_x(0,t) - \lambda^2 q(0,t)\big) \\ - e^{-i\lambda}\big(q_{xx}(1,t) + i\lambda q_x(1,t) - \lambda^2 q(1,t)\big).$$

We already know all boundary values on the right at position $x = 1$, i.e.

$$h_0(t) = q(1,t),$$
$$h_1(t) = q_x(1,t),$$
$$h_2(t) = q_{xx}(1,t).$$

Then, let

$$g_0(t) = q(0,t),$$
$$g_1(t) = q_x(0,t),$$
$$g_2(t) = q_{xx}(0,t).$$

Substituting these into the above *Q-equation* we get

$$\left[\frac{\partial}{\partial t} - i\lambda^3\right]\widehat{q}(\lambda;t) = \left(g_2(t) + i\lambda g_1(t) - \lambda^2 g_0(t)\right) - e^{-i\lambda}\left(h_2(t) + i\lambda h_1(t) - \lambda^2 h_0(t)\right). \tag{2.8}$$

5. Note that we already calculated the period T Fourier expansions of the boundary terms on the right in step 2. Now we calculate the period T Fourier expansions of the boundary terms on the left and of $\widehat{q}(\lambda;t)$. First, for $w = \frac{2\pi}{T}$, denote

$$\begin{aligned} G_j^{(0)} &:= F_{ser}[g_0](j), \\ G_j^{(1)} &:= F_{ser}[g_1](j), \\ G_j^{(2)} &:= F_{ser}[g_2](j), \\ q_j(\lambda) &:= F_{ser}[\widehat{q}(\lambda;\cdot)](j). \end{aligned}$$

Then, by the Fourier series representation for a periodic function in equation 4.11 from (Chaparro and Akan, 2018),

$$\begin{aligned} g_0(t) &= \sum_{j\in\mathbb{Z}} G_j^{(0)} e^{ij\omega t}, \\ g_1(t) &= \sum_{j\in\mathbb{Z}} G_j^{(1)} e^{ij\omega t}, \\ g_2(t) &= \sum_{j\in\mathbb{Z}} G_j^{(2)} e^{ij\omega t}, \\ \widehat{q}(\lambda;t) &= \sum_{j\in\mathbb{Z}} q_j(\lambda) e^{ij\omega t}, \\ \Rightarrow \frac{\partial}{\partial t}\widehat{q}(\lambda;t) &= \sum_{j\in\mathbb{Z}} (ij\omega) q_j(\lambda) e^{ij\omega t}. \end{aligned}$$

We substitute the period T Fourier expansions into the *Q-equation* (2.8):

$$\sum_{j\in\mathbb{Z}} e^{ij\omega t}(ij\omega - i\lambda^3)q_j(\lambda) = \sum_{j\in\mathbb{Z}} e^{ij\omega t}\Big(G_j^{(2)} + i\lambda G_j^{(1)} - \lambda^2 G_j^{(0)} \\ - e^{-i\lambda}\left(H_j^{(2)} + i\lambda H_j^{(1)} - \lambda^2 H_j^{(0)}\right)\Big).$$

Then, by Corollary 1, $\forall j \in \mathbb{Z}$

$$\begin{aligned}&(ij\omega - i\lambda^3)q_j(\lambda)\\ &= G_j^{(2)} + i\lambda G_j^{(1)} - \lambda^2 G_j^{(0)} - e^{-i\lambda}(H_j^{(2)} + i\lambda H_j^{(1)} - \lambda^2 H_j^{(0)}).\end{aligned} \tag{2.9}$$

We want to say that the above equation is zero for all λ such that $(ij\omega - i\lambda^3) = 0$. However, we should first check that $q_j(\lambda)$ is finite for any such λ.

Lemma 3. *Suppose $q_j(\lambda)$ is a function of Fourier coefficients defined by*

$$q_j(\lambda) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \widehat{q}(\lambda;t)\, e^{-ij\omega t}dt, \quad \forall\lambda \in \mathbb{C}, \forall j \in \mathbb{Z}$$

where

$$\widehat{q}(\lambda;t) = \int_0^1 e^{-i\lambda x} q(x,t)dx,$$

T is a period of $q(x,t)$ and $\omega = \frac{2\pi}{T}$. Then, $\forall\lambda \in \mathbb{C}, \forall j \in \mathbb{Z}$

$$|q_j(\lambda)| < \infty.$$

The proof of the above lemma can be found in Appendix 13. Note that by the above lemma, $|q_j(\lambda)|$ is finite and equation (2.9) holds

$\forall \lambda \in \mathbb{C}$. In particular, it holds for those λ for which the left-hand side of (2.9) is zero, i.e. for those λ for which $ij\omega - i\lambda^3 = 0$, which simplifies the equation to

$$G_j^{(2)} + i\lambda G_j^{(1)} - \lambda^2 G_j^{(0)} = e^{-i\lambda}\left(H_j^{(2)} + i\lambda H_j^{(1)} - \lambda^2 H_j^{(0)}\right). \quad (2.10)$$

For each $j \in \mathbb{Z} \setminus \{0\}$, there are three such λ: $\lambda_j, \alpha\lambda_j, \alpha^2\lambda_j$, where $\lambda_j = \sqrt[3]{j\omega}, \alpha = e^{\frac{2\pi}{3}i}$. Denote

$$N_j(\lambda) := e^{-i\lambda}(H_j^{(2)} + i\lambda H_j^{(1)} - \lambda^2 H_j^{(0)})$$

as our known data. Now applying maps $\lambda \to \lambda_j, \lambda \to \alpha\lambda_j, \lambda \to \alpha^2\lambda_j$ to equation (2.10) gives a system of three linear equations with three unknowns:

$$\begin{pmatrix} 1 & i\lambda_j & -\lambda_j^2 \\ 1 & i\alpha\lambda_j & -\alpha^2\lambda_j^2 \\ 1 & i\alpha^2\lambda_j & -\alpha\lambda_j^2 \end{pmatrix} \begin{pmatrix} G_j^{(2)} \\ G_j^{(1)} \\ G_j^{(0)} \end{pmatrix} = \begin{pmatrix} N_j(\lambda_j) \\ N_j(\alpha\lambda_j) \\ N_j(\alpha^2\lambda_j) \end{pmatrix}. \quad (2.11)$$

We seek the period T Fourier coefficients of all the boundary values on the left, i.e. $G_j^{(0)}, G_j^{(1)}, G_j^{(2)}$. We seek the unknowns using system (2.11).

Remark: This might not work if the system is singular at $\lambda_j, \alpha\lambda_j, \alpha^2\lambda_j$ as defined above. To check that, one could find zeros of the matrix using a numerical root-finding algorithm based on the principal argument. If the system is singular, we cannot proceed further, and the result is that the given initial non-local boundary value problem does not have a time-periodic or asymptotically time-periodic solution!

We proceed under the assumption that system (2.11) is not singular. The full solution to the system can be found in Appendix 14. Based on the solution, $\forall j \in \mathbb{Z} \setminus \{0\}$

$$\begin{aligned}
G_j^{(0)} &= \frac{N_j(\lambda_j)(\alpha^2-\alpha) + N_j(\alpha\lambda_j)(1-\alpha^2) + N_j(\alpha^2\lambda_j)(\alpha-1)}{-3(\alpha^2-\alpha)\lambda_j^2}, \\
G_j^{(1)} &= \frac{N_j(\lambda_j)(\alpha^2-\alpha) + N_j(\alpha\lambda_j)(\alpha-1) + N_j(\alpha^2\lambda_j)(1-\alpha^2)}{3i\lambda_j(\alpha^2-\alpha)}, \\
G_j^{(2)} &= \frac{N_j(\lambda_j) + N_j(\alpha\lambda_j) + N_j(\alpha^2\lambda_j)}{3}.
\end{aligned}$$

Now, we seek $G_j^{(0)}, G_j^{(1)}, G_j^{(2)}$ for $j = 0$. For $j = 0$, equation (2.9) becomes

$$-i\lambda^3 q_0(\lambda) = G_0^{(2)} + i\lambda G_0^{(1)} - \lambda^2 G_0^{(0)} - e^{-i\lambda}\left(H_0^{(2)} + i\lambda H_0^{(1)} - \lambda^2 H_0^{(0)}\right).$$

Letting $\lambda = 0$ and differentiating with respect to λ multiple times, we get

$$\begin{aligned}
G_0^{(2)} &= H_0^{(2)}, \\
G_0^{(1)} &= H_0^{(1)} - H_0^{(2)}, \\
G_0^{(0)} &= \frac{1}{2}H_0^{(2)} - H_0^{(1)} + H_0^{(0)}.
\end{aligned}$$

The full solution can be found in Appendix 15. Finally, we have found $G_j^{(0)}, G_j^{(1)}, G_j^{(2)}$ $\forall j \in \mathbb{Z}$. Now, we can reconstruct all the boundary terms on the left, i.e.

$$\begin{aligned}
q(0,t) &= g_0(t) \\
&= \sum_{j\in\mathbb{Z}} G_j^{(0)} e^{ij\omega t} \\
q_x(0,t) &= g_1(t) \\
&= \sum_{j\in\mathbb{Z}} G_j^{(1)} e^{ij\omega t}, \\
q_{xx}(0,t) &= g_2(t) \\
&= \sum_{j\in\mathbb{Z}} G_j^{(2)} e^{ij\omega t}.
\end{aligned}$$

6. We use the usual *Q-equation* to obtain an expression for $q_j(\lambda)$ which holds $\forall j \in \mathbb{Z}$:

$$q_j(\lambda) = \frac{G_j^{(2)} + i\lambda G_j^{(1)} - \lambda^2 G_j^{(0)} - e^{-i\lambda}\left(H_j^{(2)} + i\lambda H_j^{(1)} - \lambda^2 H_j^{(0)}\right)}{ij\omega - i\lambda^3}.$$

7. Now, we can use Fourier inversions to obtain $q(x,t)$, the solution to **Problem** (2.3). By the Fourier series representation for a periodic function in equation 4.11 from (Chaparro and Akan, 2018),

$$\widehat{q}(\lambda;t) = \sum_{j\in\mathbb{Z}} e^{ij\omega t} q_j(\lambda).$$

Now using Fourier Transform Inversion Theorem from (Braaksma, 1966),

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{q}(\lambda;t) e^{i\lambda x} d\lambda.$$

8. We evaluate $q(x,0)$ to get the initial condition $Q(x)$ for the time-periodic solution $q(x,t)$:

$$\begin{aligned}
Q(x) :&= q(x,0) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{q}(\lambda;0)e^{i\lambda x}d\lambda \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\sum_{j\in\mathbb{Z}} q_j(\lambda)d\lambda.
\end{aligned}$$

   We proceed under the assumption that $Q(x) \neq U(x)$.

9. We set up another Initial Non-local Boundary Value Problem with the solution $v(x,t)$ such that $v(x,t) = u(x,t) - q(x,t)$. First, note that the PDE stays the same:

$$\begin{aligned}
v_t &= u_t - q_t \\
&= -(u_{xxx} - q_{xxx}) \\
&= -v_{xxx}.
\end{aligned} \tag{2.12}$$

However, the boundary and non-local conditions are now homogeneous:

$$\begin{aligned}
v(1,t) &= u(1,t) - q(u,t) \\
&= h_0(t) - h_0(t) \\
&= 0, \\
v_x(1,t) &= u_x(1,t) - q_x(u,t) \\
&= h_1(t) - h_1(t) \\
&= 0, \\
\int_0^1 K(y)v(y,t)dy &= \int_0^1 K(y)(u(y,t) - q(y,t))dy \\
&= \int_0^1 K(y)u(y,t)dy - \int_0^1 K(y)q(y,t)dy \\
&= a(t) - a(t) \\
&= 0.
\end{aligned}$$

Finally, the initial condition is given by $V(x)$:

$$\begin{aligned}
V(x) :&= v(x,0) \\
&= u(x,0) - q(x,0) \\
&= U(x) - Q(x).
\end{aligned}$$

10. We can find an explicit contour integral solution for the above problem in (Smith and Normatov, Accessed 2023-03-24):

$$v(x,t) = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} e^{i\lambda x + i\lambda^3 t}\widehat{V}(\lambda)d\lambda + \int_{\partial D^+} e^{i\lambda x + i\lambda^3 t}\zeta^+(\lambda)d\lambda \right.$$
$$\left. + \int_{\partial D^-} e^{i\lambda(x-1) + i\lambda^3 t}\zeta^-(\lambda)d\lambda\right] \tag{2.13}$$

valid $\forall x \in [0,1], \forall t \in [0,\infty)$, where

$$\zeta^-(\lambda) = -\frac{W(\lambda) + \alpha W(\alpha\lambda) + \alpha^2 W(\alpha^2\lambda)}{\widehat{K}(\lambda) + \alpha\widehat{K}(\alpha\lambda) + \alpha^2\widehat{K}(\alpha^2\lambda)},$$
$$\zeta^+(\lambda) = \widehat{V}(\lambda) - e^{-i\lambda}\frac{W(\lambda) + \alpha W(\alpha\lambda) + \alpha^2 W(\alpha^2\lambda)}{\widehat{K}(\lambda) + \alpha\widehat{K}(\alpha\lambda) + \alpha^2\widehat{K}(\alpha^2\lambda)},$$

and

$$W(\lambda) = \int_0^1 K(y)e^{i\lambda y}\widehat{V}(\lambda; y, 1)dy,$$
$$\widehat{K}(\lambda) = \int_0^1 K(y)e^{-i\lambda(1-y)}dy,$$

and where the domains

$$D^{\pm} = \{\lambda \in \mathbb{C} : \Re(-i\lambda^3) < 0 \text{ and } \pm\Im(\lambda) > 0\}$$

have positively-oriented boundary.

11. Finally, we can solve **Problem** (1.1):

$$
\begin{aligned}
u(x,t) &= q(x,t) + v(x,t) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{q}(\lambda;t)e^{i\lambda x}d\lambda \\
&\quad + \frac{1}{2\pi}\Big[\int_{-\infty}^{\infty} e^{i\lambda x+i\lambda^3 t}\widehat{V}(\lambda)d\lambda + \int_{\partial D^+} e^{i\lambda x+i\lambda^3 t}\zeta^+(\lambda)d\lambda \\
&\qquad + \int_{\partial D^-} e^{i\lambda(x-1)+i\lambda^3 t}\zeta^-(\lambda)d\lambda\Big].
\end{aligned}
$$

We hope to prove that the above solution to **Problem** (1.1) is asymptotically time-periodic, which is equivalent to the error term $v(x,t)$ decaying as $t\to\infty$, which is exactly the statement of Conjecture 4 below. We use asymptotic expansions in order to re-write contour integrals that constitute $v(x,t)$. However, that is still to be shown in the following chapter.

2.3 Formulating a Conjecture

Continuing step 11 above, we can formulate a conjecture that will allow us to state that the solution to **Problem** (1.1) is indeed asymptotically time-periodic.

Conjecture 4. *Consider an Initial Non-local Boundary Value Problem with* *the Stokes equation that satisfies an explicit initial condition* (2.14.IC), *homogeneous boundary conditions* (2.14.BC1), (2.14.BC2) *and a homogeneous non-local condition* (2.14.NC):

$$v_t(x,t) + v_{xxx}(x,t) = 0 \qquad (x,t) \in [0,1] \times [0,\infty), \tag{2.14.PDE}$$

$$v(x,0) = V(x) \qquad x \in [0,1], \tag{2.14.IC}$$

$$v(1,t) = 0 \qquad t \in [0,\infty), \tag{2.14.BC1}$$

$$v_x(1,t) = 0 \qquad t \in [0,\infty), \tag{2.14.BC2}$$

$$\int_0^1 K(y)v(y,t)dy = 0 \qquad t \in [0,\infty), \tag{2.14.NC}$$

where $K(y)$ is a known continuously differentiable function, and $V(x)$ is a known functon, whose second derivative is absolutely continuous. Then, the solution to the above problem, $v(x,t)$, decays as $t \to \infty$. Moreover, the rate of decay is $\mathcal{O}\left(\frac{1}{t}\right)$.

A partial proof of the above conjectue can be found in the following chapter.

Chapter 3

Analyzing $v(x,t)$ for large t

3.1 Defining a Partial Proof

Here, we provide a partial proof of Conjecture 4. Recall the equation for $v(x,t)$ as stated in expression (2.13) and defined in step 10:

$$v(x,t) = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} e^{i\lambda x + i\lambda^3 t}\widehat{V}(\lambda)d\lambda + \int_{\partial D^+} e^{i\lambda x + i\lambda^3 t}\zeta^+(\lambda)d\lambda \right.$$
$$\left. + \int_{\partial D^-} e^{i\lambda(x-1) + i\lambda^3 t}\zeta^-(\lambda)d\lambda\right].$$

First, we re-write the above equation for $v(x,t)$:

$$v(x,t) = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} e^{i\lambda x + i\lambda^3 t}\widehat{v_0}(\lambda)d\lambda + \int_{\partial D_1^+} e^{i\lambda x + i\lambda^3 t}\zeta^+(\lambda)d\lambda \right.$$
$$\left. + \int_{\partial D_2^-} e^{i\lambda(x-1) + i\lambda^3 t}\zeta^-(\lambda)d\lambda + \int_{\partial D_3^-} e^{i\lambda(x-1) + i\lambda^3 t}\zeta^-(\lambda)d\lambda\right], \tag{3.1}$$

where

$$\begin{aligned} D_1^+ &= D^+, \\ D_2^- &= \{\lambda \in \mathbb{C} : \lambda \in D^- \text{and } \Re(\lambda) < 0\}, \\ D_3^- &= \{\lambda \in \mathbb{C} : \lambda \in D^- \text{and } \Re(\lambda) > 0\}. \end{aligned}$$

Note that equation (3.1) now consists of three contour integrals and a real integral. We say the proof is partial because while it is necessary to show the decay of all integrals in order to prove Conjecture 4, we will only analyze the last contour integral around the boundary ∂D_3^- as $t \to \infty$. That is primarily because studying the entire solution $v(x,t)$ requires more time. A more thorough discussion on this will be given in the final Chapter. In order to keep the notational burden to a minimum let

$$v_3(x,t) := \int_{\partial D_3^-} e^{i\lambda(x-1)+i\lambda^3 t} \zeta^-(\lambda) d\lambda.$$

Our aim is to show that for large $t, v_3(x,t) = \mathcal{O}(\frac{1}{t})$. First, we state the following useful lemma, whose proof can be found in Appendix 17.

Lemma 5. *Suppose $v_3 : [0,1] \times [0,\infty) \to \mathbb{C}$ is a function defined by*

$$v_3(x,t) = \int_{\partial D_3^-} e^{i\lambda(x-1)+i\lambda^3 t} \zeta^-(\lambda) d\lambda \qquad \forall x \in [0,1], t \geq 0,$$

where

$$\zeta^-(b) = -\frac{W(b) + \alpha W(\alpha b) + \alpha^2 W(\alpha^2 b)}{\widehat{K}(b) + \alpha \widehat{K}(\alpha b) + \alpha^2 \widehat{K}(\alpha^2 b)},$$

and

$$
\begin{aligned}
W(b) &= \int_0^1 K(y) e^{iby} \widehat{V}(b; y, 1) dy, \\
\widehat{K}(b) &= \int_0^1 K(y) e^{-ib(1-y)} dy, \\
\widehat{V}(b; y, 1) &= \int_y^1 e^{-ibx} V(x) dx, \\
\alpha &= e^{i\frac{2\pi}{3}},
\end{aligned}
$$

and

$$
D_3^- = \{\lambda \in \mathbb{C} : \Re(-i\lambda^3) < 0, -\Im(\lambda) > 0 \text{ and } \Re(\lambda) > 0\}.
$$

Then, $\forall x \in [0,1], t > 0$

$$
\begin{aligned}
v_3(x,t) = &-R_2(t; -a, a) \\
&- \lim_{b\to\infty} R_1(t; -b, -a) - \lim_{b\to\infty} R_3(t; a, b) \\
&+ \lim_{b\to\infty} \left(\frac{\phi_1(-b)}{-i3b^2 t} e^{ib^3 t} \right) + \lim_{b\to\infty} \left(\frac{\phi_3(b)}{-i3b^2 t} e^{-ib^3 t} \right),
\end{aligned}
$$

where $\phi_1, \phi_3, R_1, R_3,$ *and* R_2 *are as defined in Lemmas 6, 7, 8, 9, and 10 respectively.*

Using Lemma 5, the equation for $v_3(x,t)$ simplifies to

$$
\begin{aligned}
v_3(x,t) = &-R_2(t; -a, a) \\
&- \lim_{b\to\infty} R_1(t; -b, -a) - \lim_{b\to\infty} R_3(t; a, b) \\
&+ \lim_{b\to\infty} \left(\frac{\phi_1(-b)}{-i3b^2 t} e^{ib^3 t} \right) + \lim_{b\to\infty} \left(\frac{\phi_3(b)}{-i3b^2 t} e^{-ib^3 t} \right).
\end{aligned} \tag{3.2}
$$

At this point, we need to define the following lemmas, whose proofs can also be found in Appendix A.

Lemma 6. *Suppose $\phi_1 : \mathbb{R} \to \mathbb{C}$ is a function defined by*

$$\phi_1(b) = -\zeta^-(-b)e^{-i(x-1)b}$$

$\forall b \in \mathbb{R}, x \in [0,1]$, *where*

$$\zeta^-(b) = -\frac{W(b) + \alpha W(\alpha b) + \alpha^2 W(\alpha^2 b)}{\widehat{K}(b) + \alpha \widehat{K}(\alpha b) + \alpha^2 \widehat{K}(\alpha^2 b)},$$

and

$$\begin{aligned}
W(b) &= \int_0^1 K(y)e^{iby}\widehat{V}(b;y,1)dy, \\
\widehat{K}(b) &= \int_0^1 K(y)e^{-ib(1-y)}dy, \\
\widehat{V}(b;y,1) &= \int_y^1 e^{-ibx}V(x)dx, \\
\alpha &= e^{i\frac{2\pi}{3}}.
\end{aligned}$$

Then,

$$\lim_{b \to -\infty} \phi_1(b) = 0.$$

uniformly in $x \in [0,1]$.

Lemma 7. *Suppose $\phi_3 : \mathbb{R} \to \mathbb{C}$ is a function defined by*

$$\phi_3(b) = \zeta^-(be^{-i\frac{\pi}{3}})e^{i(x-1)be^{-i\frac{\pi}{3}}}e^{-i\frac{\pi}{3}}$$

$\forall b \in \mathbb{R}, x \in [0,1]$, *where*

$$\zeta^{-}(b) = -\frac{W(b) + \alpha W(\alpha b) + \alpha^2 W(\alpha^2 b)}{\widehat{K}(b) + \alpha \widehat{K}(\alpha b) + \alpha^2 \widehat{K}(\alpha^2 b)},$$

and

$$\begin{aligned} W(b) &= \int_0^1 K(y) e^{iby} \widehat{V}(b; y, 1) dy, \\ \widehat{K}(b) &= \int_0^1 K(y) e^{-ib(1-y)} dy, \\ \alpha &= e^{i\frac{2\pi}{3}}. \end{aligned}$$

Then,

$$\lim_{b \to \infty} \phi_3(b) = 0$$

uniformly in $x \in [0,1]$.

Lemma 8. *Suppose $R_1 : \mathbb{R} \to \mathbb{C}$ is the function defined by*

$$R_1(t; -b, -a) = \int_{-b}^{-a} \frac{d}{dk} \left(\frac{\phi_1(k)}{-i3k^2 t} \right) e^{-ik^3 t} dk \qquad \forall t \geq 0$$

where ϕ_1 is defined as in Lemma 6, has bounded total variation on the interval $(-\infty, -a]$, has its dependence on $x \in [0,1]$ suppressed, and $-\infty < -b < -a < 0$. Then, $\forall -a < 0$

$$\lim_{b \to \infty} R_1(t; -b, -a) = o\left(\frac{1}{t}\right)$$

uniformly in x as $t \to \infty$.

Lemma 9. *Suppose $R_3 : \mathbb{R} \to \mathbb{C}$ is the function defined by*

$$R_3(t;a,b) = \int_a^b \frac{d}{dk}\left(\frac{\phi_3(k)}{-i3k^2t}\right)e^{-ik^3t}dk \qquad \forall t \geq 0$$

where ϕ_3 is defined as in Lemma 7, has bounded total variation on the interval $[a,\infty)$, has its dependence on $x \in [0,1]$ suppressed, and $0 > a > b > \infty$. Then, $\forall a > 0$

$$\lim_{b\to\infty} R_3(t;a,b) = o\left(\frac{1}{t}\right)$$

uniformly in x as $t \to \infty$.

Lemma 10. *Suppose $R_2 : \mathbb{R} \to \mathbb{C}$ is the function defined by*

$$R_2(t;-a,a) = \int_{-a}^{a} \frac{d}{dk}\left(\frac{\phi_2(k)}{t\frac{\pi a^2}{2}e^{-i(\frac{\pi k}{6a}+\frac{\pi}{6})}}\right)e^{ita^3e^{-i(\frac{\pi k}{6a}+\frac{\pi}{6})}}dk \qquad \forall t \geq 0$$

where $\infty < -a < a < \infty$, and ϕ_2 is defined as in definitions A.3, has bounded total variation on the interval $[-a,a]$, and has its dependence on $x \in [0,1]$ suppressed. Then, $\forall a \in \mathbb{R}$

$$R_2(t;-a,a) = o\left(\frac{1}{t}\right)$$

uniformly in x as $t \to \infty$.

Now, using Lemmas 8, 10 and 9 we know that all the remainder terms in equation (3.2) decay like $o\left(\frac{1}{t}\right)$ as $t \to \infty$. Using Lemma 6, we know that

$$\lim_{b\to\infty}(\phi_1(-b)) = \lim_{b\to-\infty}(\phi_1(b)) = 0.$$

Then, as $t \to \infty$,

$$\begin{aligned}
\lim_{b\to\infty}\left(\frac{\phi_1(-b)}{-i3b^2 t}e^{ib^3 t}\right) &\leq \frac{1}{t}\lim_{b\to\infty}\left(\frac{\phi_1(-b)}{-i3b^2}\right)\lim_{b\to\infty} e^{ib^3 t} \\
&\leq \frac{1}{t}\lim_{b\to\infty}\left(\frac{\phi_1(-b)}{-i3b^2}\right)\sup_{b\in(-a,-\infty)}|e^{ib^3 t}| \\
&\leq \frac{1}{t}\lim_{b\to\infty}\left(\frac{\phi_1(-b)}{-i3b^2}\right) \\
&= \mathcal{O}\left(\frac{1}{t}\right).
\end{aligned}$$

Similarly, using Lemma 7

$$\lim_{b\to\infty}\left(\frac{\phi_3(b)}{-i3b^2 t}e^{-ib^3 t}\right) = \mathcal{O}\left(\frac{1}{t}\right).$$

Hence, equation (3.2) is equivalent to

$$\begin{aligned}
v_3(x,t) &= o\left(\frac{1}{t}\right) + o\left(\frac{1}{t}\right) + o\left(\frac{1}{t}\right) + \mathcal{O}\left(\frac{1}{t}\right) + \mathcal{O}\left(\frac{1}{t}\right) \\
&= \mathcal{O}\left(\frac{1}{t}\right).
\end{aligned}$$

Hence, the above contour integral around ∂D_3^- decays at a rate $\frac{1}{t}$ as non-negative $t \to \infty$.

Having proven that one part of the contour integral (3.1) decays, it is reasonable to expect that proving the rest will not be an issue. Even though this is yet to be shown in future research, we have partially proven Conjecture 4 and we have good evidence to expect that the rest can be proven as well.

Chapter 4

Discussion of the results

We managed to adapt the *Q-equation* method to find the solution to the IBVP with the Stokes equation, inhomogeneous time-periodic non-local and boundary conditions, and an explicit initial condition describing the state of the system at an initial time, i.e. $t = 0$. In particular, we considered Problem 1.1 with two boundary conditions (1.1.BC1), (1.1.BC2) and one non-local condition (1.1.NC). We found the solution to this problem by setting up another problem with the same data and PDE, but with the assumption that the solution is time-periodic, which necessitated an implicit initial condition that aligned with the solution. As explained in the introductory chapter, the assumption of time-periodicity of the solution, in turn, introduced an error, which is the difference between the solutions to the two problems. We expected this error term, which we called $v(x,t)$, to decay in time, which led to the formulation of Conjecture 4. While we did not manage to prove the conjecture entirely, we managed to prove part of it, which is sufficient evidence to expect that had we had more time, we could have proven the conjecture for the remaining contour integrals in the expression (3.1) for $v(x,t)$. Notably, the contour integral $v_3(x,t)$ around the boundary ∂D_3^- decays at a rate $\mathcal{O}(\frac{1}{t})$ as $t \to \infty$.

4.1 Future Projects

There are a number of points for further research on this topic:

1. We need to check that the zeros of the *Q-equation* matrix in steps 3 and 5 of the Methodology chapter do not overlap with the three λ maps we used to create systems (2.8) and (2.11). As mentioned earlier, we can do that by using a numerical root-finding algorithm based on the principal argument.

2. We need to prove that the remaining two contour integrals and a real integral in the equation (3.1) for $v(x,t)$ decay as $t \to \infty$. Moreover, it would be interesting to find out if the rate of decay is at least $\mathcal{O}(\frac{1}{t})$ as well.

3. Moreover, we can try to adapt the theory of asymptotic expansions for the Fourier type integrals as introduced in (Erdélyi, 1956) to our problems in order to show a faster decay of the error term for large t.

4. Finally, it would be interesting to solve other similar Initial Non-Local Boundary Value Problems with two or three 3 non-local conditions respectively.

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Appendix A

Useful Lemmas

In this chapter, I state and provide proof of some useful lemmas and sub-lemmas used in the capstone.

Corollary 11 (same as Corollary 1)**.** *If* $\forall x \in [-b, b]$*,*

$$\sum_{j\in\mathbb{Z}} e^{ij\pi x/b}\alpha_j = \sum_{j\in\mathbb{Z}} e^{ij\pi x/b}\beta_j.$$

Then, $\forall j \in \mathbb{Z}, \alpha_j = \beta_j$*.*

Proof. Let $E_j(x) := e^{ij\pi x/b}$. Using this notation,

$$\sum_{j\in\mathbb{Z}} E_j(x)\alpha_j = \sum_{j\in\mathbb{Z}} E_j(x)\beta_j.$$

But then, $\forall k \in \mathbb{Z}$,

$$\begin{aligned} \langle \sum_{j\in\mathbb{Z}} \alpha_j E_j, E_k \rangle &= \langle \sum_{j\in\mathbb{Z}} \beta_j E_j, E_k \rangle \\ \implies \sum_{j\in\mathbb{Z}} \alpha_j \langle E_j, E_k \rangle &= \sum_{j\in\mathbb{Z}} \beta_j \langle E_j, E_k \rangle. \end{aligned}$$

By the orthogonality of the Fourier series basis functions, all terms but $j = k$ in each of these series are 0. So,

$$\alpha_k \langle E_k, E_k \rangle = \beta_k \langle E_k, E_k \rangle.$$

By the orthogonality of the Fourier series basis functions, $\langle E_k, E_k \rangle \neq 0$. Therefore, $\alpha_k = \beta_k$. $\square$

Lemma 12 (same as Lemma 2). *Suppose $Q_j(\lambda)$ is a function of Fourier coefficients defined by*

$$Q_j(\lambda) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_0^1 e^{i\lambda y} K(y) \widehat{q}(\lambda; t, y, 1) dy \, e^{-ijwt} dt, \quad \forall \lambda \in \mathbb{C}, \forall j \in \mathbb{Z}$$

where

$$\widehat{q}(\lambda; t, y, 1) = \int_y^1 e^{-i\lambda x} q(x,t) dx,$$

T is a period of $q(x,t)$, $w = \frac{2\pi}{T}$ and $K(y)$ is a known function for $y \in [0,1]$. Then, $\forall \lambda \in \mathbb{C}, \forall j \in \mathbb{Z}$

$$|Q_j(\lambda)| < \infty.$$

Proof. First, we show that $|\widehat{q}(\lambda;t,y,1)|$ is finite for any $\lambda \in \mathbb{C}$ and for $y \in (0,1)$. Suppose $\lambda = u + iv$ for some $u, v \in \mathbb{R}$. Then,

$$\begin{aligned}
|\widehat{q}(\lambda;t,y,1)| \leq& |\widehat{q}(\lambda;t,0,1)| \\
=& \left| \int_0^1 e^{-i\lambda x} q(x,t)dx \right| \\
=& \left| \int_0^1 e^{-i(u+iv)x} q(x,t)dx \right| \\
=& \left| \int_0^1 e^{-iux} e^{vx} q(x,t)dx \right| \\
\leq& \int_0^1 |e^{-iux}||e^{vx}||q(x,t)|dx \quad \text{by Theorem 3.1.4} \\
=& \int_0^1 e^{vx}|q(x,t)|dx \quad \text{since } |e^{-iux}| = 1 \text{ and } e^{vx} \geq 0 \\
\leq& e^{v} \max_{x\in(0,1)} |q(x,t)|(1-0) \quad \text{by Theorem 3.3.1} \\
<& \infty \quad \text{because } q \text{ is continuous}
\end{aligned}$$

Note that we have also shown that $\widehat{q}(\lambda;t,0,1)$ is finite for any $\lambda \in \mathbb{C}$. This result will be useful later. Now, we also need to show that $\int_0^1 e^{i\lambda y} K(y)\widehat{q}(\lambda;t,y,1)dy$ is finite. Using the same argument as above,

$$\begin{aligned}
&\left| \int_0^1 e^{i\lambda y} K(y)\widehat{q}(\lambda;t,y,1)dy \right| \\
&= \left| \int_0^1 e^{i(u+iv)y} K(y)\widehat{q}(\lambda;t,y,1)dy \right| \\
&\leq \int_0^1 |e^{iuy}||e^{-vy}||K(y)||\widehat{q}(\lambda;t,y,1)|dy \quad \text{by Theorem 3.1.4} \\
&\leq (1-0) \max_{y\in(0,1)} |e^{-vy}| \max_{y\in(0,1)} |K(y)| \max_{y\in(0,1)} |\widehat{q}(\lambda;t,y,1)| \\
&= \max_{y\in(0,1)} |K(y)| \max_{y\in(0,1)} |\widehat{q}(\lambda;t,y,1)| \quad \text{since } \max_{y\in(0,1)} |e^{-vy}| = 1 \\
&< \infty \quad \text{by earlier argument and because } K \text{ is continuous}
\end{aligned}$$

Finally, we show that $Q_j(\lambda)$ is finite:

$$
\begin{aligned}
|Q_j(\lambda)| =& \Big| \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_0^1 e^{i\lambda y} K(y) \widehat{q}(\lambda; t, y, 1) dy \, e^{-ijwt} dt \Big| \\
\leq & \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \Big| \int_0^1 e^{i\lambda y} K(y) \widehat{q}(\lambda; t, y, 1) dy \Big| |e^{-ijwt}| dt \qquad \text{by Theorem 3.1.4} \\
= & \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \Big| \int_0^1 e^{i\lambda y} K(y) \widehat{q}(\lambda; t, y, 1) dy \Big| dt \qquad \text{since } |e^{-ijwt}| = 1 \\
\leq & \frac{1}{T} \Big(\frac{T}{2} + \frac{T}{2} \Big) \max_{t \in (-\frac{T}{2}, \frac{T}{2})} \int_0^1 e^{i\lambda y} K(y) \widehat{q}(\lambda; t, y, 1) dy \\
\leq & \max_{t \in (-\frac{T}{2}, \frac{T}{2})} \int_0^1 e^{i\lambda y} K(y) \widehat{q}(\lambda; t, y, 1) dy \\
< & \infty \qquad \text{by earlier argument}
\end{aligned}
$$

$\square$

Lemma 13 (same as Lemma 3). *Suppose $q_j(\lambda)$ is a function of Fourier coefficients defined by*

$$
q_j(\lambda) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \widehat{q}(\lambda; t) \, e^{-ijwt} dt, \quad \forall \lambda \in \mathbb{C}, \forall j \in \mathbb{Z}
$$

where

$$
\widehat{q}(\lambda; t) = \int_0^1 e^{-i\lambda x} q(x, t) dx,
$$

T is a period of $q(x,t)$ and $w = \frac{2\pi}{T}$. Then, $\forall \lambda \in \mathbb{C}, \forall j \in \mathbb{Z}$

$$
|q_j(\lambda)| < \infty.
$$

Proof. We show that $q_j(\lambda)$ is finite for any $\lambda \in \mathbb{C}$:

$$\begin{aligned}
|q_j(\lambda)| =& \left|\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \widehat{q}(\lambda;t)\, e^{-ijwt} dt\right| \\
\leq& \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |\widehat{q}(\lambda;t)||e^{-ijwt}| dt \quad \text{by Theorem 3.1.4} \\
=& \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |\widehat{q}(\lambda;t)| dt \quad \text{since } |e^{-ijwt}| = 1 \\
\leq& \frac{1}{T}\left(\frac{T}{2}+\frac{T}{2}\right) \max_{t\in(-\frac{T}{2},\frac{T}{2})} \widehat{q}(\lambda;t) \\
\leq& \max_{t\in(-\frac{T}{2},\frac{T}{2})} \widehat{q}(\lambda;t) \\
\leq& e^{Im(\lambda)} \max_{x\in(0,1)} |q(x,t)| \quad \text{by earlier argument} \\
<& \infty \quad \text{because } q \text{ is continuous in both } x \text{ and } t
\end{aligned}$$

□

Solution 14. *First, we simplify the system to*

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & \alpha-1 & \alpha^2-1 \\ 0 & \alpha^2-1 & \alpha-1 \end{pmatrix} \begin{pmatrix} G_j^{(2)} \\ i\lambda_j G_j^{(1)} \\ -\lambda_j^2 G_j^{(0)} \end{pmatrix} = \begin{pmatrix} N_j(\lambda_j) \\ N_j(\alpha\lambda_j) - N_j(\lambda_j) \\ N_j(\alpha^2\lambda_j) - N_j(\lambda_j) \end{pmatrix}.$$

Then, its determinant is

$$\Delta(\lambda_j) = 3(\alpha^2 - \alpha).$$

Using Cramer's rule we get:

$$\Delta_{G_j^{(0)}}(\lambda_j) = N_j(\lambda_j)(\alpha^2 - \alpha) + N_j(\alpha\lambda_j)(1 - \alpha^2) + N_j(\alpha^2\lambda_j)(\alpha - 1),$$

$$\Delta_{G_j^{(1)}}(\lambda_j) = N_j(\lambda_j)(\alpha^2 - \alpha) + N_j(\alpha\lambda_j)(\alpha - 1) + N_j(\alpha^2\lambda_j)(1 - \alpha^2),$$

$$\Delta_{G_j^{(2)}}(\lambda_j) = (\alpha^2 - \alpha)(N_j(\lambda_j) + N_j(\alpha\lambda_j) + N_j(\alpha^2\lambda_j)).$$

Hence, $\forall j \in \mathbb{Z} \setminus \{0\}$

$$\begin{aligned} G_j^{(0)} &= \frac{N_j(\lambda_j)(\alpha^2 - \alpha) + N_j(\alpha\lambda_j)(1 - \alpha^2) + N_j(\alpha^2\lambda_j)(\alpha - 1)}{-3(\alpha^2 - \alpha)\lambda_j^2}, \\ G_j^{(1)} &= \frac{N_j(\lambda_j)(\alpha^2 - \alpha) + N_j(\alpha\lambda_j)(\alpha - 1) + N_j(\alpha^2\lambda_j)(1 - \alpha^2)}{3i\lambda_j(\alpha^2 - \alpha)}, \\ G_j^{(2)} &= \frac{N_j(\lambda_j) + N_j(\alpha\lambda_j) + N_j(\alpha^2\lambda_j)}{3}. \end{aligned}$$

Solution 15. *For* $j = 0$*, equation* (2.9) *becomes*

$$-i\lambda^3 q_0(\lambda) = G_0^{(2)} + i\lambda G_0^{(1)} - \lambda^2 G_0^{(0)} - e^{-i\lambda}\left(H_0^{(2)} + i\lambda H_0^{(1)} - \lambda^2 H_0^{(0)}\right).$$

Letting $\lambda = 0$ *we get*

$$G_0^{(2)} = H_0^{(2)}.$$

Differentiating with respect to λ*:*

$$\begin{aligned} -3i\lambda^2 q_0(\lambda) - i\lambda^3 q_0'(\lambda) = iG_0^{(1)} - 2\lambda G_0^{(0)} \\ + ie^{-i\lambda}\left(H_0^{(2)} + i\lambda H_0^{(1)} - \lambda^2 H_0^{(0)}\right) \\ - e^{-i\lambda}\left(iH_0^{(1)} - 2\lambda H_0^{(0)}\right). \end{aligned}$$

Letting $\lambda = 0$, we get

$$G_0^{(1)} = H_0^{(1)} - H_0^{(2)}.$$

Differentiating again with respect to λ:

$$\begin{aligned}-6i\lambda q_0(\lambda) - 6i\lambda^2 q_0'(\lambda) - i\lambda^3 q_0''(\lambda)& \\ = -2G_0^{(0)} + e^{-i\lambda}\big(H_0^{(2)} + i\lambda H_0^{(1)} - \lambda^2 H_0^{(0)}\big)& \\ + 2ie^{-i\lambda}\big(iH_0^{(1)} - 2\lambda H_0^{(0)}\big) - e^{-i\lambda}\big(-2H_0^{(0)}\big).&\end{aligned}$$

Letting $\lambda = 0$, we get

$$\begin{aligned}0 &= -2G_0^{(0)} + H_0^{(2)} - 2H_0^{(1)} + 2H_0^{(0)} \\ \Rightarrow G_0^{(0)} &= \frac{1}{2}H_0^{(2)} - H_0^{(1)} + H_0^{(0)}.\end{aligned}$$

Finally, we have found $G_j^{(0)}, G_j^{(1)}, G_j^{(2)}\ \forall j = 0$.

Definition 16. *(**Path Integral or Contour Integral**) Suppose that γ is a path over a closed interval $[a,b]$ and that f is a continuous complex-valued function defined on the graph of γ. The **path** or **contour integral** of f on γ is defined as:*

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

(Adapted from Asmar and Grafakos, 2018).

Lemma 17 (same as Lemma 5)**.** *Suppose $v_3 : \mathbb{R}^2 \to \mathbb{C}$ is a function defined by*

$$v_3(x,t) = \int_{\partial D_3^-} e^{i\lambda(x-1)+i\lambda^3 t}\zeta^-(\lambda)d\lambda \qquad \forall x \in [0,1], t \geq 0$$

where

$$\zeta^{-}(b) = -\frac{W(b) + \alpha W(\alpha b) + \alpha^2 W(\alpha^2 b)}{\widehat{K}(b) + \alpha \widehat{K}(\alpha b) + \alpha^2 \widehat{K}(\alpha^2 b)}$$

and

$$\begin{aligned}
W(b) &= \int_0^1 K(y) e^{iby} \widehat{V}(b; y, 1) dy \\
\widehat{K}(b) &= \int_0^1 K(y) e^{-ib(1-y)} dy \\
\widehat{V}(b; y, 1) &= \int_y^1 e^{-ibx} V(x) dx \\
\alpha &= e^{i\frac{2\pi}{3}}
\end{aligned}$$

and

$$D_3^- = \{\lambda \in \mathbb{C} : Re(-i\lambda^3) < 0, -Im(\lambda) > 0 \text{ and } Re(\lambda) > 0\}$$

Then, $\forall x \in [0,1], t > 0$

$$\begin{aligned}
v_3(x,t) = &-R_2(t; -a, a) \\
&- \lim_{b\to\infty} R_1(t; -b, -a) - \lim_{b\to\infty} R_3(t; a, b) \\
&+ \lim_{b\to\infty} \left(\frac{\phi_1(-b)}{-i3b^2 t} e^{ib^3 t} \right) + \lim_{b\to\infty} \left(\frac{\phi_3(b)}{-i3b^2 t} e^{-ib^3 t} \right)
\end{aligned}$$

where $\phi_1, \phi_3, R_1, R_3,$ *and* R_2 *are as defined in Lemmas 6, 7, 8, 9, and 10 respectively.*

Proof. We start by rewriting this contour integral using Countour Path Definition 16. We use the following parameterization $\gamma(k)$ of the contour path around ∂D_3^-:

$$\gamma(k) = \begin{cases} -k & \text{if } k \in (-\infty, -a] \\ ae^{-i(\frac{\pi k}{6a}+\frac{\pi}{6})} & \text{if } k \in (-a, a) \\ ke^{-i\frac{\pi}{3}} & \text{if } k \in [a, \infty) \end{cases} \tag{A.1}$$

where a is a positive real number near 0. Then,

$$\begin{aligned} v_3(x,t) &= \int_{\partial D_3^-} e^{i\lambda^3 t}\zeta^-(\lambda)e^{i\lambda(x-1)}d\lambda \\ &= \int_{-\infty}^{-a} e^{-ik^3t}\phi_1(k)dk \\ &\quad + \int_{-a}^{a} e^{ita^3e^{-i(\frac{\pi k}{6a}+\frac{\pi}{6})}}\phi_2(k)dk \\ &\quad + \int_{a}^{\infty} e^{-ik^3t}\phi_3(k)dk \\ &= \lim_{b\to\infty}\int_{-b}^{-a} e^{-ik^3t}\phi_1(k)dk \\ &\quad + \int_{-a}^{a} e^{ita^3e^{-i(\frac{\pi k}{6a}+\frac{\pi}{6})}}\phi_2(k)dk \\ &\quad + \lim_{b\to\infty}\int_{a}^{b} e^{-ik^3t}\phi_3(k)dk \end{aligned} \tag{A.2}$$

where

$$\begin{aligned} \phi_1(k) &:= -\zeta^-(-k)e^{-i(x-1)k} \\ \phi_2(k) &:= \zeta^-(ae^{-i(\frac{\pi k}{6a}+\frac{\pi}{6})})e^{i(x-1)ae^{-i(\frac{\pi k}{6a}+\frac{\pi}{6})}}(-i)\frac{\pi}{6a}ae^{-i(\frac{\pi k}{6a}+\frac{\pi}{6})} \\ \phi_3(k) &:= \zeta^-(ke^{-i\frac{\pi}{3}})e^{i(x-1)ke^{-i\frac{\pi}{3}}}e^{-i\frac{\pi}{3}} \end{aligned} \tag{A.3}$$

In order to asymptotically analyze the above real integrals for large t, we integrate each of them by parts:

$$\begin{aligned}
\lim_{b\to\infty}\int_{-b}^{-a} e^{-ik^3t}\phi_1(k)dk &= \lim_{b\to\infty}\int_{-b}^{-a}\phi_1(k)\frac{1}{-i3k^2t}\frac{d}{dk}(e^{-ik^3t})dk\\
&= \lim_{b\to\infty}\Big[\frac{\phi_1(k)}{-i3k^2t}e^{-ik^3t}\Big]_{-b}^{-a} - \lim_{b\to\infty}\int_{-b}^{-a}\frac{d}{dk}\Big(\frac{\phi_1(k)}{-i3k^2t}\Big)e^{-ik^3t}dk\\
&= \frac{\phi_1(-a)}{-i3a^2t}e^{ia^3t} - \lim_{b\to\infty}\Big(\frac{\phi_1(-b)}{-i3b^2t}e^{ib^3t}\Big) - \lim_{b\to\infty}R_1(t;-b,-a)
\end{aligned}$$

where

$$R_1(t;-b,-a) := \int_{-b}^{-a}\frac{d}{dk}\Big(\frac{\phi_1(k)}{-i3k^2t}\Big)e^{-ik^3t}dk$$

Similarly,

$$\begin{aligned}
&\lim_{b\to\infty}\int_{a}^{b} e^{-ik^3t}\phi_3(k)dk\\
&\qquad = \lim_{b\to\infty}\Big(\frac{\phi_3(b)}{-i3b^2t}e^{-ib^3t}\Big) - \frac{\phi_3(a)}{-i3a^2t}e^{-ia^3t} - \lim_{b\to\infty}R_3(t;a,b)
\end{aligned}$$

where

$$R_3(t;a,b) := \int_{a}^{b}\frac{d}{dk}\Big(\frac{\phi_3(k)}{-i3k^2t}\Big)e^{-ik^3t}dk$$

Lastly,

$$\begin{aligned}
\int_{-a}^{a} e^{ita^3e^{-i(\frac{\pi k}{6a}+\frac{\pi}{6})}}\phi_2(k)dk &= \int_{-a}^{a}\phi_2(k)\frac{1}{t\frac{\pi a^2}{2}e^{-i(\frac{\pi k}{6a}+\frac{\pi}{6})}}\frac{d}{dk}\Big[e^{ita^3e^{-i(\frac{\pi k}{6a}+\frac{\pi}{6})}}\Big]dk\\
&= \left[\frac{\phi_2(k)}{t\frac{\pi a^2}{2}e^{-i(\frac{\pi k}{6a}+\frac{\pi}{6})}}e^{ita^3e^{-i(\frac{\pi k}{6a}+\frac{\pi}{6})}}\right]_{-a}^{a}\\
&\quad - \int_{-a}^{a}\frac{d}{dk}\left(\frac{\phi_2(k)}{t\frac{\pi a^2}{2}e^{-i(\frac{\pi k}{6a}+\frac{\pi}{6})}}\right)e^{ita^3e^{-i(\frac{\pi k}{6a}+\frac{\pi}{6})}}dk\\
&= \frac{\phi_2(a)}{t\frac{\pi a^2}{2}(-1)}e^{-ita^3} - \frac{\phi_2(-a)}{t\frac{\pi a^2}{2}}e^{ita^3} - R_2(t;-a,a)
\end{aligned}$$

where

$$R_2(t;-a,a) := \int_{-a}^{a} \frac{d}{dk}\left(\frac{\phi_2(k)}{t\frac{\pi a^2}{2}e^{-i(\frac{\pi k}{6a}+\frac{\pi}{6})}}\right) e^{ita^3 e^{-i(\frac{\pi k}{6a}+\frac{\pi}{6})}}dk$$

Now, we can rewrite equation (A.2) using expansions of the integrals that we have accomplished above using integration by parts:

$$\begin{aligned}
v_3(x,t) &= \frac{\phi_1(-a)}{-i3a^2t}e^{ia^3t} - \lim_{b\to\infty}\left(\frac{\phi_1(-b)}{-i3b^2t}e^{ib^3t}\right) - \lim_{b\to\infty} R_1(t;-b,-a) \\
&+ \frac{\phi_2(a)}{t\frac{\pi a^2}{2}(-1)}e^{-ita^3} - \frac{\phi_2(-a)}{t\frac{\pi a^2}{2}}e^{ita^3} - R_2(t;-a,a) \\
&+ \lim_{b\to\infty}\left(\frac{\phi_3(b)}{-i3b^2t}e^{-ib^3t}\right) - \frac{\phi_3(a)}{-i3a^2t}e^{-ia^3t} - \lim_{b\to\infty} R_3(t;a,b) \\
&= \sum\phi - R_2(t;-a,a) \\
&- \lim_{b\to\infty} R_1(t;-b,-a) - \lim_{b\to\infty} R_3(t;a,b) \\
&+ \lim_{b\to\infty}\left(\frac{\phi_1(-b)}{-i3b^2t}e^{ib^3t}\right) + \lim_{b\to\infty}\left(\frac{\phi_3(b)}{-i3b^2t}e^{-ib^3t}\right)
\end{aligned} \tag{A.4}$$

where

$$\sum\phi := \frac{\phi_1(-a)}{-i3a^2t}e^{ia^3t} - \frac{\phi_2(-a)}{t\frac{\pi a^2}{2}}e^{ita^3} + \frac{\phi_2(a)}{t\frac{\pi a^2}{2}(-1)}e^{-ita^3} - \frac{\phi_3(a)}{-i3a^2t}e^{-ia^3t}$$

We want to show that all of the terms in the above equation (A.4) of $v_3(x,t)$ decay as $t\to\infty$ and that the entire equation is $\mathcal{O}(\frac{1}{t})$. We do that by analyzing every term in the equation individually. Using equations for ϕ_1, ϕ_2 and ϕ_3 in the definitions (A.3), we evaluate

$$
\begin{aligned}
\phi_1(-a) &= -\zeta^-(a)e^{i(x-1)a} \\
\phi_2(-a) &= -i\frac{\pi}{6}\zeta^-(a)e^{i(x-1)a} \\
\phi_2(a) &= -i\frac{\pi}{6}\zeta^-(ae^{-i\frac{\pi}{3}})e^{i(x-1)ae^{-i\frac{\pi}{3}}}e^{-i\frac{\pi}{3}} \\
\phi_3(a) &= \zeta^-(ae^{-i\frac{\pi}{3}})e^{i(x-1)ae^{-i\frac{\pi}{3}}}e^{-i\frac{\pi}{3}}
\end{aligned}
$$

Then, the equation for $\sum\phi$ is equal 0:

$$
\begin{aligned}
\sum\phi &= e^{-ia^3t}\Big[\frac{\phi_2(a)}{t\frac{\pi a^2}{2}(-1)} - \frac{\phi_3(a)}{-i3a^2t}\Big] + e^{ia^3t}\Big[\frac{\phi_1(-a)}{-i3a^2t} - \frac{\phi_2(-a)}{t\frac{\pi a^2}{2}}\Big] \\
&= e^{-ia^3t}\Big[i\frac{\pi}{6}\zeta^-(ae^{-i\frac{\pi}{3}})e^{i(x-1)ae^{-i\frac{\pi}{3}}}e^{-i\frac{\pi}{3}}\frac{2}{t\pi a^2} \\
&\quad + \zeta^-(ae^{-i\frac{\pi}{3}})e^{i(x-1)ae^{-i\frac{\pi}{3}}}e^{-i\frac{\pi}{3}}\frac{1}{i3a^2t}\Big] \\
&\quad + e^{ia^3t}\Big[\zeta^-(a)e^{i(x-1)a}\frac{1}{i3a^2t} + i\frac{\pi}{6}\zeta^-(a)e^{i(x-1)a}\frac{2}{t\pi a^2}\Big] \\
&= e^{-ia^3t}\Big[\frac{1}{3a^2t}\zeta^-(ae^{-i\frac{\pi}{3}})e^{i(x-1)ae^{-i\frac{\pi}{3}}}e^{-i\frac{\pi}{3}}\Big(i+\frac{1}{i}\Big)\Big] \\
&\quad + e^{ia^3t}\Big[\zeta^-(a)e^{i(x-1)a}\frac{1}{3a^2t}\Big(i+\frac{1}{i}\Big)\Big] \\
&= 0
\end{aligned}
$$

Now using this result equation (A.4) simplifies to

$$
\begin{aligned}
v_3(x,t) &= -R_2(t;-a,a) \\
&\quad - \lim_{b\to\infty} R_1(t;-b,-a) - \lim_{b\to\infty} R_3(t;a,b) \\
&\quad + \lim_{b\to\infty}\left(\frac{\phi_1(-b)}{-i3b^2t}e^{ib^3t}\right) + \lim_{b\to\infty}\left(\frac{\phi_3(b)}{-i3b^2t}e^{-ib^3t}\right)
\end{aligned}
$$

□

Lemma 18 (same as Lemma 6)**.** *Suppose $\phi_1 : \mathbb{R} \to \mathbb{C}$ is a function defined by*

$$\phi_1(b) = -\zeta^-(-b)e^{-i(x-1)b}$$

$\forall b \in \mathbb{R}, x \in [0,1]$ *where*

$$\zeta^-(b) = -\frac{W(b) + \alpha W(\alpha b) + \alpha^2 W(\alpha^2 b)}{\widehat{K}(b) + \alpha \widehat{K}(\alpha b) + \alpha^2 \widehat{K}(\alpha^2 b)}$$

and

$$\begin{aligned} W(b) &= \int_0^1 K(y)e^{iby}\widehat{V}(b;y,1)dy \\ \widehat{K}(b) &= \int_0^1 K(y)e^{-ib(1-y)}dy \\ \widehat{V}(b;y,1) &= \int_y^1 e^{-ibx}V(x)dx \\ \alpha &= e^{i\frac{2\pi}{3}}. \end{aligned}$$

Then,

$$\lim_{b\to-\infty} \phi_1(b) = 0.$$

uniformly in $x \in [0,1]$.

Proof. We start by expanding $\phi_1(b)$ using its definition in the statement of the lemma.

$$
\begin{aligned}
\lim_{b\to-\infty}\phi_1(b) &= \lim_{b\to-\infty}\Big(-\zeta^-(-b)e^{-i(x-1)b}\Big) \\
&= \lim_{b\to-\infty}\Big(-\zeta^-(-b)\Big)\lim_{b\to-\infty}e^{-i(x-1)b} \\
&\le \lim_{b\to\infty}\Big(-\zeta^-(b)\Big)\sup_{b\in(-a,-\infty)}|e^{-i(x-1)b}| \\
&= \lim_{b\to\infty}\Big(-\zeta^-(b)\Big)\times 1 \\
&= \lim_{b\to\infty}\Big(\frac{W(b)+\alpha W(\alpha b)+\alpha^2 W(\alpha^2 b)}{\widehat{K}(b)+\alpha\widehat{K}(\alpha b)+\alpha^2\widehat{K}(\alpha^2 b)}\Big) \qquad \text{(A.5)}
\end{aligned}
$$

We want to show that the numerator's decay rate is greater than the rate of decay of the denominator as $b\to\infty$. We start with the asymptotic analysis of the denominator.

Analysis of $\widehat{K}(b)$ as $b\to\infty$.

We start by analyzing the first term. We integrate by parts to reveal the decay rates of the terms that comprise the integral:

$$
\begin{aligned}
\widehat{K}(b) &= \int_0^1 K(y)e^{-ib(1-y)}dy \\
&= \Big[\frac{1}{ib}e^{-ib(1-y)}K(y)\Big]_{y=0}^{y=1} - \frac{1}{ib}\int_0^1 e^{-ib(1-y)}K'(y)dy \\
&= \frac{1}{ib}K(1) - \frac{1}{ib}e^{-ib}K(0) - \frac{1}{ib}\int_0^1 K'(y)e^{-ib(1-y)}dy \\
&= \mathcal{O}\Big(\frac{1}{b}\Big) \qquad \text{(A.6)}
\end{aligned}
$$

because as $b\to\infty$, $e^{-ib(1-y)}$ is bounded by 1 and the decay rate of the last term is $\mathcal{O}\Big(\frac{1}{b}\Big)$ times the decay rate of the integral, which decays the same as $\widehat{K}(b)$.

Analysis of $\widehat{K}(\alpha b)$ as $b\to\infty$.

Now, we move on to analyzing the second term of the denominator.

$$
\begin{aligned}
\widehat{K}(\alpha b) &= \int_0^1 K(y)e^{-i\alpha b(1-y)}dy \\
&= \left[\frac{-1}{i\alpha b}e^{-i\alpha b(1-y)}K(y)\right]_{t=0}^{t=1} + \frac{1}{i\alpha b}\int_0^1 e^{-i\alpha b(1-y)}K'(y)dy \\
&= \frac{-1}{i\alpha b}K(1) + \frac{1}{i\alpha b}e^{-i\alpha b}K(0) + \frac{1}{i\alpha b}\int_0^1 K'(y)e^{-i\alpha b(1-y)}dy \\
&= \mathcal{O}\left(\frac{e^{-i\alpha b}}{b}\right) \qquad \text{(A.7)}
\end{aligned}
$$

because $e^{-i\alpha b}$ blows up as $b \to \infty$.

Analysis of $\widehat{K}(\alpha^2 b)$ as $b \to \infty$.

Finally, we asymptotically analyze the last term.

$$
\begin{aligned}
\widehat{K}(\alpha^2 b) &= \int_0^1 K(y)e^{-i\alpha^2 b(1-y)}dy \\
&= \left[\frac{-1}{i\alpha^2 b}e^{-i\alpha^2 b(1-y)}K(y)\right]_{y=0}^{y=1} + \frac{1}{i\alpha^2 b}\int_0^1 e^{-i\alpha^2 b(1-y)}K'(y)dy \\
&= \frac{-1}{i\alpha^2 b}K(1) + \frac{1}{i\alpha^2 b}e^{-i\alpha^2 b}K(0) + \frac{1}{i\alpha^2 b}\int_0^1 K'(y)e^{-i\alpha^2 b(1-y)}dy \\
&= \mathcal{O}\left(\frac{1}{b}\right) \qquad \text{(A.8)}
\end{aligned}
$$

because $e^{-i\alpha b}$ decays as $b \to \infty$ and the decay rate of the last term is $\mathcal{O}\left(\frac{1}{b}\right)$ times the decay rate of the integral, which decays as fast as $\widehat{K}(\alpha^2 b)$ itself.

Hence, using (A.6), (A.7), (A.8) we are able to analyze the rate of decay of the denominator in A.5:

$$
\begin{aligned}
\widehat{K}(b) + \alpha\widehat{K}(\alpha b) + \alpha^2\widehat{K}(\alpha^2 b) &= \mathcal{O}\left(\frac{1}{b}\right) + \mathcal{O}\left(\frac{e^{-i\alpha b}}{b}\right) + \mathcal{O}\left(\frac{1}{b}\right) \\
&= \mathcal{O}\left(\frac{e^{-i\alpha b}}{b}\right) \qquad \text{(A.9)}
\end{aligned}
$$

Now we move on to the asymptotic analysis of the numerator in A.5. We analyze each term of the numerator individually as $b \to \infty$. We start with the asymptotic analysis of $W(b)$.

Analysis of $W(b)$ as $b \to \infty$.

Note that e^{-ib} is bounded by 1 as $b \to \infty$. In fact, it is bounded for all $b \in \mathbb{R}$. Now, we need a preliminary result that will be used later:

$$\begin{aligned}\int_y^1 e^{-ib(x-y)}V(x)dx &= \left[\frac{-1}{ib}e^{-ib(x-y)}V(x)\right]_{x=y}^{x=1} + \frac{1}{ib}\int_y^1 e^{-ib(x-y)}V'(x)dx \\ &= \frac{-1}{ib}e^{-ib(1-y)}V(1) + \frac{1}{ib}V(y) + \frac{1}{ib}\int_y^1 e^{-ib(x-y)}V'(x)dx\end{aligned} \tag{A.10}$$

Then,

$$\begin{aligned}W(b) &= \int_0^1 K(y)e^{iby}\widehat{V}(b;y,1)dy \\ &= \int_0^1 K(y)e^{iby}\int_y^1 e^{-ibx}V(x)dxdy \\ &= \int_0^1 K(y)\int_y^1 e^{-ib(x-y)}V(x)dxdy\end{aligned}$$

using A.10

$$\begin{aligned}&= \int_0^1 K(y)\Big[\frac{-1}{ix}e^{-ib(1-y)}V(1) + \frac{1}{ib}V(y) \\ &\quad + \frac{1}{ib}\int_y^1 e^{-ib(x-y)}V'(x)dx\Big]dy \\ &= \frac{-1}{ib}V(1)\int_0^1 K(y)e^{-ib(1-y)}dy + \frac{1}{ib}\int_0^1 K(y)V(y)dy \\ &\quad + \frac{1}{ib}\int_0^1 K(y)\int_y^1 e^{-ib(x-y)}V'(x)dxdy\end{aligned}$$

using A.6 we know that the first integral is equal to $\widehat{K}(b) = \mathcal{O}\left(\frac{1}{b}\right)$, thus the first term is $\mathcal{O}(\frac{1}{b^2})$; however, the second term is $\mathcal{O}(\frac{1}{b})$; and, the third term is $\mathcal{O}(\frac{1}{b})$ times the decay rate of the integral, which decays exactly the same as $W(b)$ itself. Hence,

$$W(b) = \mathcal{O}\left(\frac{1}{b}\right) \tag{A.11}$$

Analysis of $W(\alpha b)$ as $b \to \infty$.

Now,we move onto analyzing $W(\alpha b)$. First we need a few preliminary results. Note that $\mathcal{O}(e^{-i\alpha b})$ blows up as $b \to \infty$. Moreover,

$$\begin{aligned}
&\int_y^1 e^{-i\alpha b(x-y)} V(x) dx \\
&\qquad = \left[\frac{-1}{i\alpha b} e^{-i\alpha b(x-y)} V(x)\right]_{x=y}^{x=1} + \frac{1}{i\alpha b}\int_y^1 e^{-i\alpha b(x-y)} V'(x) dx \\
&\qquad = \frac{-1}{i\alpha b} e^{-i\alpha b(1-y)} V(1) + \frac{1}{i\alpha b} V(y) + \frac{1}{i\alpha b}\int_y^1 e^{-i\alpha b(x-y)} V'(x) dx
\end{aligned} \tag{A.12}$$

Then,

$$\begin{aligned}
W(\alpha b) &= \int_0^1 K(y) e^{\alpha b y} \widehat{V}(\alpha b; y, 1) dy \\
&= \int_0^1 K(y) e^{i\alpha b y} \int_y^1 e^{-i\alpha b x} V(x) dx dy \\
&= \int_0^1 K(y) \int_y^1 e^{-i\alpha b(x-y)} V(x) dx dy
\end{aligned}$$

using A.12,

$$
\begin{aligned}
&= \int_0^1 K(y)\Big[\frac{-1}{i\alpha b}e^{-i\alpha b(1-y)}V(1) + \frac{1}{i\alpha b}V(y) \\
&\quad + \frac{1}{i\alpha b}\int_y^1 e^{-i\alpha b(x-y)}V'(x)dx\Big]dy \\
&= \frac{-1}{i\alpha b}V(1)\int_0^1 K(y)e^{-i\alpha b(1-y)}dy + \frac{1}{i\alpha b}\int_0^1 K(y)V(y)dy \\
&\quad + \frac{1}{i\alpha b}\int_0^1 K(y)\int_y^1 e^{-i\alpha b(x-y)}V'(x)dxdy
\end{aligned}
$$

using A.7, we know that the first integral blows up at a rate $\mathcal{O}\left(\frac{e^{-i\alpha b}}{b}\right)$, so when it is multiplied by $\left(\frac{-1}{i\alpha b}\right)$, the entire first term is $\mathcal{O}\left(\frac{e^{-i\alpha b}}{b^2}\right)$, which is the most dominant term as the second term is only $\mathcal{O}\left(\frac{1}{b}\right)$, while the last term is $\mathcal{O}\left(\frac{1}{b}\right)$ times the decay rate of the third integral, which decays just as fast as $W(\alpha b)$ itself. Hence,

$$W(\alpha b) = \mathcal{O}\left(\frac{e^{-i\alpha b}}{b^2}\right) \tag{A.13}$$

Analysis of $W(\alpha^2 b)$ as $b \to \infty$.

Finally, we analyze the last term in the numerator, $W(\alpha^2 b)$. Note that this time the main exponential term inside the integral, $e^{-i\alpha^2 b}$ decays as $b \to \infty$. Once again, first, we need the following result:

$$
\begin{aligned}
&\int_y^1 e^{-i\alpha^2 b(x-y)}V(x)dx \\
&\qquad = \left[\frac{-1}{i\alpha^2 b}e^{-i\alpha^2 b(x-y)}V(x)\right]_{x=y}^{x=1} + \frac{1}{i\alpha^2 b}\int_y^1 e^{-i\alpha^2 b(x-y)}V'(x)dx \\
&\qquad = \frac{-1}{i\alpha^2 b}e^{-i\alpha^2 b(1-y)}V(1) + \frac{1}{i\alpha^2 b}V(y) + \frac{1}{i\alpha^2 b}\int_y^1 e^{-i\alpha^2 b(x-y)}V'(x)dx
\end{aligned} \tag{A.14}
$$

Then,

$$\begin{aligned}
W(\alpha^2 b) &= \int_0^1 K(y) e^{i\alpha^2 by} \widehat{V}(\alpha^2 b; y, 1) dy \\
&= \int_0^1 K(y) e^{i\alpha^2 by} \int_y^1 e^{-i\alpha^2 bx} V(x) dx dy \\
&= \int_0^1 K(y) \int_y^1 e^{-i\alpha^2 b(x-y)} V(x) dx dy
\end{aligned}$$

using A.14,

$$\begin{aligned}
&= \int_0^1 K(y) \Big[\frac{-1}{i\alpha^2 b} e^{-i\alpha^2 b(1-y)} V(1) + \frac{1}{i\alpha^2 b} V(y) \\
&\quad + \frac{1}{i\alpha^2 b} \int_y^1 e^{-i\alpha^2 b(x-y)} V'(x) dx \Big] dy \\
&= \frac{-1}{i\alpha^2 b} V(1) \int_0^1 K(y) e^{-i\alpha^2 b(1-y)} dy + \frac{1}{i\alpha^2 b} \int_0^1 K(y) V(y) dy \\
&\quad + \frac{1}{i\alpha^2 b} \int_0^1 K(y) \int_y^1 e^{-i\alpha^2 b(x-y)} V'(x) dx dy
\end{aligned}$$

using A.8, we know that the first integral decays up at a rate $\mathcal{O}\left(\frac{1}{b}\right)$, so when it is multiplied by $\left(\frac{-1}{i\alpha^2 b}\right)$, the entire first term is $\mathcal{O}\left(\frac{1}{b^2}\right)$; the second term is $\mathcal{O}\left(\frac{1}{b}\right)$, and the last term is $\mathcal{O}\left(\frac{1}{b}\right)$ times the decay rate of the third integral, which decays just as fast as $W(\alpha^2 b)$. Hence,

$$W(\alpha^2 b) = \mathcal{O}\left(\frac{1}{b}\right) \tag{A.15}$$

Finally, we can analyze the entire numerator in A.5 and use A.11, A.13 and A.15 to get

$$W(b) + \alpha W(\alpha b) + \alpha^2 W(\alpha^2 b) = \mathcal{O}\left(\frac{1}{b}\right) + \mathcal{O}\left(\frac{e^{-i\alpha b}}{b^2}\right) + \mathcal{O}\left(\frac{1}{b}\right) = \mathcal{O}\left(\frac{e^{-i\alpha b}}{b^2}\right) \tag{A.16}$$

Now, we move on to the analysis of the entire fraction in A.5 as $b \to \infty$. Using results A.9 and A.16, we get

$$\lim_{b\to\infty} \phi_1(b) \le \lim_{b\to\infty} \left(\frac{W(b) + \alpha W(\alpha b) + \alpha^2 W(\alpha^2 b)}{\widehat{K}(b) + \alpha \widehat{K}(\alpha b) + \alpha^2 \widehat{K}(\alpha^2 b)} \right)$$
$$= \lim_{b\to\infty} \left(\frac{\mathcal{O}\left(\frac{e^{-i\alpha b}}{b^2}\right)}{\mathcal{O}\left(\frac{e^{-i\alpha b}}{b}\right)} \right)$$
$$= \lim_{b\to\infty} \mathcal{O}\left(\frac{1}{b}\right)$$
$$= 0 \tag{A.17}$$

And with this result, Lemma 18 is proven. □

Lemma 19 (same as Lemma 7)**.** *Suppose $\phi_3 : \mathbb{R} \to \mathbb{C}$ is a function defined by*

$$\phi_3(b) = \zeta^-(be^{-i\frac{\pi}{3}})e^{i(x-1)be^{-i\frac{\pi}{3}}} e^{-i\frac{\pi}{3}} \qquad \forall b \in \mathbb{R}, x \in [0,1]$$

where

$$\zeta^-(b) = -\frac{W(b) + \alpha W(\alpha b) + \alpha^2 W(\alpha^2 b)}{\widehat{K}(b) + \alpha \widehat{K}(\alpha b) + \alpha^2 \widehat{K}(\alpha^2 b)}$$

and

$$W(b) = \int_0^1 K(y) e^{iby} \widehat{V}(b; y, 1) dy$$
$$\widehat{K}(b) = \int_0^1 K(y) e^{-ib(1-y)} dy$$
$$\alpha = e^{i\frac{2\pi}{3}}.$$

Then,

$$\lim_{b\to\infty} \phi_3(b) = 0$$

uniformly in $x \in [0, 1]$.

Proof.

$$\begin{aligned}
\lim_{b\to\infty} \phi_3(b) &= \lim_{b\to\infty} \left(\zeta^-(be^{-i\frac{\pi}{3}})e^{i(x-1)be^{-i\frac{\pi}{3}}}e^{-i\frac{\pi}{3}}\right) \\
&\leq \lim_{b\to\infty} |\zeta^-(be^{-i\frac{\pi}{3}})e^{i(x-1)be^{-i\frac{\pi}{3}}}e^{-i\frac{\pi}{3}}| \\
&\leq \lim_{b\to\infty} |\zeta^-(be^{-i\frac{\pi}{3}})||e^{i(x-1)be^{-i\frac{\pi}{3}}}||e^{-i\frac{\pi}{3}}| \\
&= \lim_{b\to\infty} |\zeta^-(be^{-i\frac{\pi}{3}})| \lim_{b\to\infty} |e^{i(x-1)b(\frac{1}{2}-i\frac{\sqrt{3}}{2})}| \\
&= \lim_{b\to\infty} |\zeta^-(be^{-i\frac{\pi}{3}})| \lim_{b\to\infty} |e^{\frac{\sqrt{3}}{2}(x-1)b}||e^{i\frac{1}{2}(x-1)b}| \\
&= \lim_{b\to\infty} |\zeta^-(be^{-i\frac{\pi}{3}})| \lim_{b\to\infty} |e^{\frac{\sqrt{3}}{2}(x-1)b}||e^{i\frac{1}{2}(x-1)b}| \qquad \text{(A.18)}
\end{aligned}$$

Note that $(\frac{\sqrt{3}}{2}(x-1)b) \to -\infty$ as $b \to \infty$ because $(x-1) \leq 0$. Thus, $e^{\frac{\sqrt{3}}{2}(x-1)b} \to 0$ as $b \to \infty$. Now, all that remains to show is that

$$\lim_{b\to\infty} |\zeta^-(be^{-i\frac{\pi}{3}})| = 0$$

First, we realize that $e^{-i\frac{\pi}{3}} = -e^{i\frac{2\pi}{3}} = -\alpha$. Using the definition of ζ^- from the statement of the Lemma 19, we get

$$\begin{aligned}
\zeta^-(be^{-i\frac{\pi}{3}}) &= \zeta^-(-\alpha b) \\
&= -\frac{W(-\alpha b) + \alpha W(-\alpha^2 b) + \alpha^2 W(-b)}{\widehat{K}(-\alpha b) + \alpha \widehat{K}(-\alpha^2 b) + \alpha^2 \widehat{K}(-b)} \qquad \text{(A.19)}
\end{aligned}$$

We want to show that the numerator's decay rate is greater than the rate of decay of the denominator as $b \to \infty$. We start with the asymptotic analysis of the denominator.

Analysis of $\widehat{K}(-b)$ as $b \to \infty$.

We start by analyzing the first term. We integrate by parts to reveal the decay rates of the terms that comprise the integral:

$$\begin{aligned}
\widehat{K}(-b) &= \int_0^1 K(y)e^{ib(1-y)}dy \\
&= \Big[\frac{-1}{ib}e^{ib(1-y)}K(y)\Big]_{y=0}^{y=1} + \frac{1}{ib}\int_0^1 e^{ib(1-y)}K'(y)dy \\
&= \frac{-1}{ib}K(1) + \frac{1}{ib}e^{ib}K(0) + \frac{1}{ib}\int_0^1 K'(y)e^{ib(1-y)}dy \\
&= \mathcal{O}\Big(\frac{1}{b}\Big) \qquad \text{(A.20)}
\end{aligned}$$

because as $b \to \infty$, e^{ib} is bounded by 1 and the decay rate of the last term is $\mathcal{O}\left(\frac{1}{b}\right)$ times the decay rate of the integral, which decays at the same rate as $\widehat{K}(-b)$.

Analysis of $\widehat{K}(-\alpha b)$ as $b \to \infty$.

Now, we move on to analyzing the second term of the denominator.

$$\begin{aligned}
\widehat{K}(-\alpha b) &= \int_0^1 K(y)e^{i\alpha b(1-y)}dy \\
&= \Big[\frac{-1}{i\alpha b}e^{i\alpha b(1-y)}K(y)\Big]_{t=0}^{t=1} + \frac{1}{i\alpha b}\int_0^1 e^{i\alpha b(1-y)}K'(y)dy \\
&= \frac{-1}{i\alpha b}K(1) + \frac{1}{i\alpha b}e^{i\alpha b}K(0) + \frac{1}{i\alpha b}\int_0^1 K'(y)e^{i\alpha b(1-y)}dy \\
&= \mathcal{O}\Big(\frac{1}{b}\Big) \qquad \text{(A.21)}
\end{aligned}$$

because $e^{i\alpha b}$ decays as $b \to \infty$.

Analysis of $\widehat{K}(-\alpha^2 b)$ as $b \to \infty$.

Finally, we asymptotically analyze the last term.

$$
\begin{aligned}
\widehat{K}(-\alpha^2 b) &= \int_0^1 K(y) e^{i\alpha^2 b(1-y)} dy \\
&= \left[\frac{-1}{i\alpha^2 b} e^{i\alpha^2 b(1-y)} K(y)\right]_{y=0}^{y=1} + \frac{1}{i\alpha^2 b}\int_0^1 e^{i\alpha^2 b(1-y)} K'(y) dy \\
&= \frac{-1}{i\alpha^2 b} K(1) + \frac{1}{i\alpha^2 b} e^{i\alpha^2 b} K(0) + \frac{1}{i\alpha^2 b}\int_0^1 K'(y) e^{i\alpha^2 b(1-y)} dy \\
&= \mathcal{O}\left(\frac{e^{i\alpha^2 b}}{b}\right) \qquad \text{(A.22)}
\end{aligned}
$$

because $e^{i\alpha^2 b}$ blows up as $b \to \infty$ and the decay rate of the last term is $\mathcal{O}\left(\frac{1}{b}\right)$ times the decay rate of the integral, which decays as fast as $\widehat{K}(-\alpha^2 b)$ itself.

Hence, using A.20, A.21, A.22 we are able to analyze the rate of decay of the denominator in A.19:

$$
\begin{aligned}
\widehat{K}(-\alpha b) + \alpha \widehat{K}(-\alpha^2 b) + \alpha^2 \widehat{K}(-b) &= \mathcal{O}\left(\frac{1}{b}\right) + \mathcal{O}\left(\frac{e^{i\alpha^2 b}}{b}\right) + \mathcal{O}\left(\frac{1}{b}\right) \\
&= \mathcal{O}\left(\frac{e^{i\alpha^2 b}}{b}\right) \qquad \text{(A.23)}
\end{aligned}
$$

Now we move on to the asymptotic analysis of the numerator in A.19. We analyze each term of the numerator individually as $b \to \infty$. We start with the asymptotic analysis of $W(-b)$.

Analysis of $W(-b)$ as $b \to \infty$.

Note that e^{ib} is bounded by 1 as $b \to \infty$. In fact, it is bounded for all $b \in \mathbb{R}$. Now, we need a preliminary result that will be used later:

$$\begin{aligned}\int_y^1 e^{ib(x-y)} V(x)dx &= \left[\frac{1}{ib} e^{ib(x-y)} V(x)\right]_{x=y}^{x=1} - \frac{1}{ib}\int_y^1 e^{ib(x-y)} V'(x)dx \\ &= \frac{1}{ib} e^{ib(1-y)} V(1) - \frac{1}{ib} V(y) - \frac{1}{ib}\int_y^1 e^{ib(x-y)} V'(x)dx\end{aligned} \tag{A.24}$$

Then,

$$\begin{aligned}W(-b) &= \int_0^1 K(y) e^{-iby} \widehat{V}(-b; y, 1)dy \\ &= \int_0^1 K(y) e^{-iby} \int_y^1 e^{ibx} V(x)dxdy \\ &= \int_0^1 K(y) \int_y^1 e^{ib(x-y)} V(x)dxdy\end{aligned}$$

using A.24

$$\begin{aligned}&= \int_0^1 K(y) \Big[\frac{1}{ix} e^{ib(1-y)} V(1) - \frac{1}{ib} V(y) \\ &\quad - \frac{1}{ib}\int_y^1 e^{ib(x-y)} V'(x)dx\Big] dy \\ &= \frac{1}{ib} V(1) \int_0^1 K(y) e^{ib(1-y)} dy - \frac{1}{ib}\int_0^1 K(y)V(y)dy \\ &\quad - \frac{1}{ib}\int_0^1 K(y) \int_y^1 e^{ib(x-y)} V'(x)dxdy\end{aligned}$$

using A.20 we know that the first integral is equal to $\widehat{K}(-b) = \mathcal{O}\left(\frac{1}{b}\right)$, thus the first term is $\mathcal{O}(\frac{1}{b^2})$; however, the second term is $\mathcal{O}(\frac{1}{b})$; and, the third term is $\mathcal{O}(\frac{1}{b})$ times the decay rate of the integral, which decays exactly at the same rate as $W(-b)$ itself as $b \to \infty$. Hence,

$$W(-b) = \mathcal{O}\left(\frac{1}{b}\right) \tag{A.25}$$

Analysis of $W(-\alpha b)$ as $b \to \infty$.

Now,we move onto analyzing $W(-\alpha b)$. First we need a few preliminary results. Note that $\mathcal{O}(e^{i\alpha b})$ blows up as $b \to \infty$. Moreover,

$$\begin{aligned}
&\int_y^1 e^{i\alpha b(x-y)} V(x) dx \\
&\qquad = \left[\frac{1}{i\alpha b} e^{i\alpha b(x-y)} V(x)\right]_{x=y}^{x=1} - \frac{1}{i\alpha b}\int_y^1 e^{i\alpha b(x-y)} V'(x) dx \\
&\qquad = \frac{1}{i\alpha b} e^{i\alpha b(1-y)} V(1) - \frac{1}{i\alpha b} V(y) - \frac{1}{i\alpha b}\int_y^1 e^{i\alpha b(x-y)} V'(x) dx
\end{aligned} \tag{A.26}$$

Then,

$$\begin{aligned}
W(-\alpha b) &= \int_0^1 K(y) e^{-\alpha b y} \widehat{V}(-\alpha b; y, 1) dy \\
&= \int_0^1 K(y) e^{-i\alpha b y} \int_y^1 e^{i\alpha b x} V(x) dx dy \\
&= \int_0^1 K(y) \int_y^1 e^{i\alpha b(x-y)} V(x) dx dy
\end{aligned}$$

using A.26,

$$
\begin{aligned}
&= \int_0^1 K(y)\Big[\frac{1}{i\alpha b}e^{i\alpha b(1-y)}V(1) - \frac{1}{i\alpha b}V(y) \\
&\quad - \frac{1}{i\alpha b}\int_y^1 e^{i\alpha b(x-y)}V'(x)dx\Big]dy \\
&= \frac{1}{i\alpha b}V(1)\int_0^1 K(y)e^{i\alpha b(1-y)}dy - \frac{1}{i\alpha b}\int_0^1 K(y)V(y)dy \\
&\quad - \frac{1}{i\alpha b}\int_0^1 K(y)\int_y^1 e^{i\alpha b(x-y)}V'(x)dxdy
\end{aligned}
$$

using A.21, we know that the first integral decays at a rate $\mathcal{O}\left(\frac{1}{b}\right)$, so when it is multiplied by $\left(\frac{-1}{i\alpha b}\right)$, the entire first term is $\mathcal{O}\left(\frac{1}{b^2}\right)$; the second term is $\mathcal{O}\left(\frac{1}{b}\right)$, while the last term is $\mathcal{O}\left(\frac{1}{b}\right)$ times the decay rate of the third integral, which decays just as fast as $W(-\alpha b)$ itself. Hence,

$$
W(-\alpha b) = \mathcal{O}\left(\frac{1}{b}\right) \tag{A.27}
$$

Analysis of $W(-\alpha^2 b)$ as $b \to \infty$.

Finally, we analyze the last term in the numerator, $W(-\alpha^2 b)$. Note that this time the main exponential term inside the integral, $e^{i\alpha^2 b}$ blows up as $b \to \infty$. Once again, first, we need the following preliminary result:

$$
\begin{aligned}
&\int_y^1 e^{i\alpha^2 b(x-y)}V(x)dx \\
&\quad = \left[\frac{1}{i\alpha^2 b}e^{i\alpha^2 b(x-y)}V(x)\right]_{x=y}^{x=1} - \frac{1}{i\alpha^2 b}\int_y^1 e^{i\alpha^2 b(x-y)}V'(x)dx \\
&\quad = \frac{1}{i\alpha^2 b}e^{i\alpha^2 b(1-y)}V(1) - \frac{1}{i\alpha^2 b}V(y) - \frac{1}{i\alpha^2 b}\int_y^1 e^{i\alpha^2 b(x-y)}V'(x)dx
\end{aligned} \tag{A.28}
$$

Then,

$$
\begin{aligned}
W(-\alpha^2 b) &= \int_0^1 K(y) e^{-i\alpha^2 by} \widehat{V}(-\alpha^2 b; y, 1) dy \\
&= \int_0^1 K(y) e^{-i\alpha^2 by} \int_y^1 e^{i\alpha^2 bx} V(x) dx dy \\
&= \int_0^1 K(y) \int_y^1 e^{i\alpha^2 b(x-y)} V(x) dx dy
\end{aligned}
$$

using A.28,

$$
\begin{aligned}
&= \int_0^1 K(y) \Big[\frac{1}{i\alpha^2 b} e^{i\alpha^2 b(1-y)} V(1) - \frac{1}{i\alpha^2 b} V(y) \\
&\quad - \frac{1}{i\alpha^2 b} \int_y^1 e^{i\alpha^2 b(x-y)} V'(x) dx \Big] dy \\
&= \frac{1}{i\alpha^2 b} V(1) \int_0^1 K(y) e^{i\alpha^2 b(1-y)} dy - \frac{1}{i\alpha^2 b} \int_0^1 K(y) V(y) dy \\
&\quad - \frac{1}{i\alpha^2 b} \int_0^1 K(y) \int_y^1 e^{i\alpha^2 b(x-y)} V'(x) dx dy
\end{aligned}
$$

using A.22, we know that the first integral blows up at a rate $\mathcal{O}\Big(\frac{e^{i\alpha^2 b}}{b}\Big)$, so when it is multiplied by $\Big(\frac{-1}{i\alpha^2 b}\Big)$, the entire first term is $\mathcal{O}\Big(\frac{e^{i\alpha^2 b}}{b^2}\Big)$; the second term is $\mathcal{O}\Big(\frac{1}{b}\Big)$, and the last term is $\mathcal{O}\Big(\frac{1}{b}\Big)$ times the decay rate of the third integral, which decays just as fast as $W(-\alpha^2 b)$. Hence,

$$
W(-\alpha^2 b) = \mathcal{O}\Big(\frac{e^{i\alpha^2 b}}{b^2}\Big) \tag{A.29}
$$

Finally, we can analyze the entire numerator in A.19 and use A.25, A.27 and A.29 to get

$$
\begin{aligned}
W(-\alpha b) + \alpha W(-\alpha^2 b) + \alpha^2 W(-b) &= \mathcal{O}\Big(\frac{1}{b}\Big) + \mathcal{O}\Big(\frac{e^{i\alpha^2 b}}{b^2}\Big) + \mathcal{O}\Big(\frac{1}{b}\Big) \\
&= \mathcal{O}\Big(\frac{e^{i\alpha^2 b}}{b^2}\Big)
\end{aligned} \tag{A.30}
$$

Now, we move on to the analysis of the entire fraction in A.19 as $b \to \infty$. Using results A.23 and A.30, we get

$$\lim_{b\to\infty} \phi_3(b) \le \lim_{b\to\infty} \Big(\zeta^-(-\alpha b)\Big) \tag{A.31}$$

$$= \lim_{b\to\infty} \Big(- \frac{W(-\alpha b) + \alpha W(-\alpha^2 b) + \alpha^2 W(-b)}{\widehat{K}(-\alpha b) + \alpha \widehat{K}(-\alpha^2 b) + \alpha^2 \widehat{K}(-b)} \Big)$$

$$= \lim_{b\to\infty} \left(\frac{\mathcal{O}\left(\frac{e^{i\alpha^2 b}}{b^2}\right)}{\mathcal{O}\left(\frac{e^{i\alpha^2 b}}{b}\right)} \right)$$

$$= \lim_{b\to\infty} \mathcal{O}\Big(\frac{1}{b}\Big)$$

$$= 0 \tag{A.32}$$

And with this result, Lemma 19 is proven. □

Lemma 20 (same as Lemma 8)**.** *Suppose $R_1 : \mathbb{R} \to \mathbb{C}$ is the function defined by*

$$R_1(t; -b, -a) = \int_{-b}^{-a} \frac{d}{dk} \left(\frac{\phi_1(k)}{-i3k^2 t} \right) e^{-ik^3 t} dk \qquad \forall t \ge 0$$

where ϕ_1 is defined as in Lemma 6, and has its dependence on $x \in [0,1]$ suppressed, and $-\infty < -b < -a < 0$.

Then, assuming $\phi_1(k)$ has total bounded variation on the interval $(-\infty, -a]$, $\forall -a < 0$

$$\lim_{b\to\infty} R_1(t; -b, -a) = o\left(\frac{1}{t}\right)$$

uniformly in x as $t \to \infty$.

Proof. Consider

$$R_1(t; -b, -a) = \int_{-b}^{-a} \frac{d}{dk} \left(\frac{\phi_1(k)}{-i3k^2 t} \right) e^{-ik^3 t} dk$$
$$= \frac{-1}{i3t} \int_{-b}^{-a} \frac{d}{dk} \left(\frac{\phi_1(k)}{k^2} \right) e^{-ik^3 t} dk$$

Note that $k^2 \neq 0$ as $-b \leq k \leq -a < 0$. We proceed by substitution. For $k \in (-b, -a)$, let $r = -k^3$. Then, $k = -r^{1/3}$ and $\frac{dr}{dk} = -3k^2$. The latter also implies $dk = \frac{dr}{-3k^2}$. Using these substitutions we get

$$\begin{aligned}
\frac{-1}{i3t} \int_{-b}^{-a} \frac{d}{dk} \left(\frac{\phi_1(k)}{k^2} \right) e^{-ik^3 t} dk &= \frac{-1}{i3t} \int_{b^3}^{a^3} -3k^2 \frac{d}{dr} \left(\frac{\phi_1(-r^{1/3})}{r^{2/3}} \right) e^{irt} \frac{1}{-3k^2} dr \\
&= \frac{-1}{i3t} \int_{b^3}^{a^3} \frac{d}{dr} \left(\frac{\phi_1(-r^{1/3})}{r^{2/3}} \right) e^{irt} dr
\end{aligned}$$

Let

$$\tau(r) := \frac{\phi_1(-r^{1/3})}{r^{2/3}}$$

Then,

$$\begin{aligned}
\lim_{b \to \infty} R_1(t; -b, -a) &= \lim_{b^3 \to \infty} \frac{-1}{i3t} \int_{b^3}^{a^3} \frac{d}{dr} \left(\tau(r) \right) e^{irt} dr \\
&= \frac{1}{i3t} \lim_{b^3 \to \infty} \int_{a^3}^{b^3} \frac{d}{dr} \left(\tau(r) \right) e^{irt} dr
\end{aligned}$$

Our goal is to use Reimann-Lebesgue lemma on the above integral to show that the integral decays as $t \to \infty$. To use the famous lemma, we need to show that $\frac{d(\tau(r))}{dr}$ is bounded and integrable on the interval:

$$\lim_{b^3 \to \infty} \int_{a^3}^{b^3} \left| \frac{d(\tau(r))}{dr} \right| dr < \infty \tag{A.33}$$

Note that $\tau(r)$ is continuously differentiable as it is a product of smooth functions. We can rewrite it as $\tau(r) = P(r) + iG(r)$, where $P(r), G(r)$ are real-valued functions. Then,

$$\begin{aligned}
\lim_{b^3\to\infty}\int_{a^3}^{b^3}\left|\frac{d(\tau(r))}{dr}\right|dr &= \lim_{b^3\to\infty}\int_{a^3}^{b^3}\left|\frac{d(P(r)+iG(r))}{dr}\right|dr \\
&= \lim_{b^3\to\infty}\int_{a^3}^{b^3}\left|P'(r)+iG'(r)\right|dr \\
&\leq \lim_{b^3\to\infty}\int_{a^3}^{b^3}\left|P'(r)\right|+\left|iG'(r)\right|dr \\
&= \lim_{b^3\to\infty}\int_{a^3}^{b^3}\left|P'(r)\right|dr+\lim_{b^3\to\infty}\int_{a^3}^{b^3}\left|G'(r)\right|dr
\end{aligned}$$

We start by subdividing interval $[a^3, b^3)$ into subintervals

$$[r_1, r_2), [r_2, r_3), \ldots, [r_{n\_max(b)}, r_{n\_max(b)+1})$$

where $P'(r)$ changes sign $n\_max(b)$ times on the interval and $a^3 = r_1 < 2_2 < r_3 < \cdots < r_{n\_max(b)} < r_{n\_max(b)+1} = b^3$. Without Loss of Generality, assume that $P'(r) \geq 0$ on the first interval $[r_1, r_2)$. Then, $P'(r) \geq 0$ for all intervals that start with an odd index of r and $P'(r) < 0$ for all intervals that start with an even index of r. Then,

$$\begin{aligned}
\lim_{b^3\to\infty}\int_{a^3}^{b^3}\left|P'(r)\right|dr &= \lim_{b^3\to\infty}\sum_{n=1}^{n_max(b)/2}\left(\int_{r_{2n}}^{r_{2n-1}}P'(r)dr-\int_{r_{2n+1}}^{r_{2n}}P'(r)dr\right) \\
&= \lim_{b^3\to\infty}\sum_{n=1}^{n_max(b)/2}\left([P(r)]_{r_{2n}}^{r_{2n-1}}-[P(r)]_{r_{2n+1}}^{r_{2n}}\right) \\
&= \lim_{b^3\to\infty}\sum_{n=1}^{n_max(b)/2}\left(P(r_{2n-1})-P(r_{2n})-P(r_{2n})+P(r_{2n+1})\right) \\
&= \lim_{b^3\to\infty}P(r_1)-2P(r_2)+2P(r_3)-\ldots
\end{aligned}$$

Here is perhaps the biggest assumption that we have to make for the sake of moving the argument forward, which is that the above sum is bounded. This is equivalent to the assumption that $P(r)$ has total bounded variation. So assuming that the above sum is bounded, we replicate the argument to show that $\lim_{b^3\to\infty}\int_{a^3}^{b^3}|G'(r)|\,dr$ is bounded as well under the assumption of the total bounded variation of $G(r)$.

Therefore, we have shown with necessary assumptions that

$$\lim_{b^3\to\infty}\int_{a^3}^{b^3}\left|\frac{d(\tau(r))}{dr}\right|dr<\infty$$

Finally, this result allows us to apply Riemann-Lebesgue Lemma:

$$\begin{aligned}&\lim_{t\to\infty}\lim_{b^3\to\infty}\int_{a^3}^{b^3}\frac{d}{dr}\left(\tau(r)\right)e^{irt}dr=0\\ \Longrightarrow\ &\lim_{t\to\infty}\lim_{b\to\infty}\int_{-b}^{-a}\frac{d}{dk}\left(\frac{\phi_1(k)}{k^2}\right)e^{-ik^3t}dk=0\end{aligned}\tag{A.34}$$

Then,

$$\begin{aligned}\lim_{b\to\infty}R_1(t;-b,-a)&=\frac{-1}{i3t}\lim_{b\to\infty}\int_{-b}^{-a}\frac{d}{dk}\left(\frac{\phi_1(k)}{k^2}\right)e^{-ik^3t}dk\\ &=o\left(\frac{1}{t}\right)\end{aligned}\tag{A.35}$$

because $\frac{-1}{i3t}=\mathcal{O}\left(\frac{1}{t}\right)$, while $\lim_{b\to\infty}\int_{-b}^{-a}\frac{d}{dk}\left(\frac{\phi_1(k)}{k^2}\right)e^{-ik^3t}dk$ decays to 0 as real positive $t\to\infty$. □

Lemma 21 (same as Lemma 9)**.** *Suppose $R_3:\mathbb{R}\to\mathbb{C}$ is the function defined by*

$$R_3(t;a,b)=\int_a^b\frac{d}{dk}\Big(\frac{\phi_3(k)}{-i3k^2t}\Big)e^{-ik^3t}dk\qquad\forall t\geq 0$$

where ϕ_3 is defined as in Lemma 7, and has its dependence on $x\in[0,1]$ suppressed, and $0>a>b>\infty$.

Then, assuming $\phi_3(k)$ has total bounded variation on the interval $[a, \infty)$, $\forall a > 0$

$$\lim_{b\to\infty} R_3(t; a, b) = o\left(\frac{1}{t}\right)$$

uniformly in x as $t \to \infty$.

Proof. The proof of the above lemma is analogous to the proof of Lemma 20

□

Lemma 22 (same as Lemma 10)**.** *Suppose $R_2 : \mathbb{R} \to \mathbb{C}$ is the function defined by*

$$R_2(t; -a, a) = \int_{-a}^{a} \frac{d}{dk}\left(\frac{\phi_2(k)}{t\frac{\pi a^2}{2} e^{-i(\frac{\pi k}{6a}+\frac{\pi}{6})}}\right) e^{ita^3 e^{-i(\frac{\pi k}{6a}+\frac{\pi}{6})}} dk \qquad \forall t \geq 0$$

where $\infty < -a < a < \infty$, and ϕ_2 is defined as in definitions A.3, and has its dependence on $x \in [0, 1]$ suppressed.

Then, assuming $\phi_2(k)$ has total bounded variation on the interval $[-a, a]$, $\forall a \in \mathbb{R}$

$$R_2(t; -a, a) = o\left(\frac{1}{t}\right)$$

uniformly in x as $t \to \infty$.

Proof. The proof of the above lemma is analogous to the proof of Lemma 20

□