D to N maps for the heat equation in cylindrical coordinates

Michelle Ong

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Abstract

A D to N map provides the unknown boundary values given the boundary data of an initial boundary value problem. A novel method, known as the Q equation method, was introduced in 2021 by Fokas and van der Weele for constant coefficient linear evolution equations on the half line. The method was generalized by Fokas, Pelloni and Smith in 2022 to study problems on the finite interval. In the present work, we significantly expand upon these works by demonstrating applicability of the Q equation method to evolution equations with variable coefficients. Indeed, we argue that the spatial Fourier transform traditionally employed to derive the Q equation is unnecessary, and may be replaced by an an alternative integral transformation, whose inverse need not be readily discernible. To illustrate the extension, we present the D to N maps for the heat equation on a disc and an annulus.

1 Introduction

1.1 Eigenfunctions of $\mathcal{D}^n$

Let $n \in \mathbb{N}$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Then $\mathrm{e}^{\mathrm{i}\lambda x}$ is a formal eigenfunction of the formal differential operator $\mathcal{D}^n$ with formal eigenvalue λ^n. We wish to find the other formal eigenfunctions with the same eigenvalue; that is, functions which satisfy the equation $\mathcal{D}^n f = \lambda^n f$. Consider the roots of unity, $\mathrm{e}^{\mathrm{i}(\frac{2\pi}{n})x}$. When $\mathcal{D}^n$ is applied to this function, we have $\mathcal{D}^n \mathrm{e}^{\mathrm{i}(\frac{2\pi}{n})x} = (\frac{2\pi}{n})^n e^{\mathrm{i}(\frac{2\pi}{n})x}$, which is of the form that we want. Further, the eigenspace of λ^n (i.e., the space spanned by the eigenfunctions associated with the eigenvalue λ^n) has dimension n, since there are n linearly independent eigenfunctions which, by definition, span the eigenspace.

Imposing a boundary form on the domain of the operator reduces the dimension of the eigenspace by one. To see this, consider an example with $\Phi = \{\varphi \in C^\infty[0,1]\}$. Then any function v in the eigenspace of $\mathcal{D}^n$ can be expressed as a linear combination of the eigenfunctions, $v = \sum_{j=1}^n \alpha_j \mathrm{e}^{\mathrm{i}(\frac{2\pi}{j})x}$. If we imposed a boundary condition on the domain, say $\varphi(1) = 0$, then for $x = 1$, we have $\alpha_1 \mathrm{e}^{\mathrm{i}(\frac{2\pi}{1})} + \cdots + \alpha_n \mathrm{e}^{\mathrm{i}(\frac{2\pi}{n})} = 0$. But dividing by $\mathrm{e}^{\mathrm{i}(\frac{2\pi}{1})}$ allows us to express α_1 as a linear combination of the other eigenfunctions. In other words, one of the original eigenfunctions is in the span of the others, so the dimension is $n-1$.

We want to find the eigenfunctions of the operator $\mathcal{D}^n$ with eigenvalue 0. Consider a polynomial of degree $n-1$. Every term in the polynomial, when differentiated n times, becomes 0. So eigenfunctions with eigenvalue 0 are x^{n-1}, and there are n of these eigenfunctions.

1.2 Linear superposition and IBVPs

Suppose we know a function v which satisfies

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\mathcal{L}\right]v(x,t)&=0,\\ v(0,t)&=f(t),\\ v(1,t)&=g(t),\end{aligned}$$

and we want to solve the problem

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\mathcal{L}\right]q(x,t)&=0,\\ q(0,t)&=f(t),\\ q(1,t)&=g(t),\\ q(x,0)&=Q(x).\end{aligned}$$

Because the differential operators $\frac{\partial}{\partial t}$ and $\mathcal{L}$ are linear, we can use the principle of linear superposition. Let $u=q-v$. Then we have

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\mathcal{L}\right]u(x,t)&=0,\\ u(0,t)&=f(t)-f(t)=0,\\ u(1,t)&=g(t)-g(t)=0,\\ u(x,0)&=Q(x)-v(x,0).\end{aligned}$$

So the problem has been changed from one (in q) with 3 inhomogeneous equations and 1 homogeneous equation to one (in u) with 1 inhomogeneous equation and 3 homogeneous equations, which is possibly easier to solve.

Consider the following initial boundary value problem:

$$\begin{aligned}\left[\frac{\partial}{\partial t}-K\frac{\partial^2}{\partial x^2}\right]q(x,t)&=0,\\ q(0,t)&=f(t),\\ q(1,t)&=g(t),\\ q(x,0)&=Q(x).\end{aligned}$$

From section 5 of the lecture notes, we have already found a function $v(x,t)$ which satisfies

$$\begin{aligned}\left[\frac{\partial}{\partial t}-K\frac{\partial^2}{\partial x^2}\right]v(x,t)&=0,\\ v(0,t)&=f(t),\\ v(1,t)&=g(t),\\ v(x,0)&=R(x).\end{aligned}$$

Using linear superposition again, let $u=q-v$, then we have

$$\begin{aligned}\left[\frac{\partial}{\partial t}-K\frac{\partial^2}{\partial x^2}\right]u(x,t)&=0,\\ u(0,t)&=0,\\ u(1,t)&=0,\\ u(x,0)&=Q(x)-R(x).\end{aligned}$$

This is an easier problem to solve, as we have only one inhomogeneous equation. In fact, we have already found $u(x,t)$ in question 5 of problem set 3. Then $q = u + v$.

1.3 The Q equation method for 1 dimensional heat equation

Consider again the IBVP from section 1.2:

$$\begin{aligned}\left[\frac{\partial}{\partial t} - K\frac{\partial^2}{\partial x^2}\right] q(x,t) &= 0,\\ q(0,t) &= f(t),\\ q(1,t) &= g(t),\\ q(x,0) &= Q(x).\end{aligned}$$

This is a problem involving the heat equation in one spatial dimension. We will use the Q equation method to solve the D to N map for this problem; that is, we wish to find $q_x(0,t)$ and $q_x(1,t)$.

First, extend the spatial domain of q to the real line by the rule $q(x,t) = 0$ if $x \notin [0,1]$. Then we can apply the Fourier exponential transform to the PDE:

$$\begin{aligned}&\mathcal{F}[q_t(x,t)](\lambda) + K\mathcal{F}[-q_{xx}(x,t)](\lambda) = 0\\ \implies &\int_0^1 q_t(x,t)\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}x - K\int_0^1 q_{xx}(x,t)\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}x = 0,\end{aligned}$$

writing out the Fourier transform explicitly, so we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 q(x,t)\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}x - K\left(\mathrm{e}^{-\mathrm{i}\lambda x}\left(q_x(1,t) + \mathrm{i}\lambda q(1,t)\right) - \left(q_x(0,t) + \mathrm{i}\lambda q(0,t)\right) - \lambda^2\int_0^1 q(x,t)\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}x\right) = 0,$$

bringing the time derivative out of the integral and integrating by parts twice, which gives us

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[q](\lambda;t) + K\lambda^2\mathcal{F}[q](\lambda;t) = -K\left(\left(q_x(0,t) + \mathrm{i}\lambda q(0,t)\right) - \mathrm{e}^{-\mathrm{i}\lambda x}\left(q_x(1,t) + \mathrm{i}\lambda q(1,t)\right)\right).$$

This equation, which relates the Fourier transform of q to the time derivative of the Fourier transform of q, where q satisfies a given PDE and BCs (or, equivalently, a differential operator), is called a Q equation.

Next, we reduce the problem by applying the Fourier exponential series transform $\mathcal{F}_{\text{ser}}$ on the interval $[-T/2, T/2]$ of the time variable in the Q equation. While none of the functions involved are defined for $t < 0$, corollary 2.5.7 of the lecture notes shows that this is not a problem, as long as we extend the definitions from $[0,T]$ to $[-T/2, T/2]$ appropriately.

For notational simplicity, we define

$$\hat{q}(\lambda;t) := \mathcal{F}[q](\lambda;t) = \int_{-\infty}^{\infty} q(x,t)\mathrm{e}^{-\mathrm{i}\lambda x},$$

where F denotes the Fourier exponential transform and the domain of $q(\cdot,t)$ has been extended from the interval I to $(-\infty,\infty)$ as 0 everywhere outside of I.

Then the Q equation is

$$\left[\frac{\partial}{\partial t} + K\lambda^2\right]\hat{q}(\lambda;t) = -K\left(\left(\mathrm{i}\lambda f(t) + a(t)\right) - \mathrm{e}^{-\mathrm{i}\lambda}\left(\mathrm{i}\lambda g(t) + b(t)\right)\right), \tag{1}$$

in which f and g are the boundary data of the problem, and the unknown Neumann boundary values (which we seek) are denoted by

$$\begin{aligned} a(t) &:= q_x(0,t), \\ b(t) &:= q_x(1,t). \end{aligned}$$

We denote

$$\begin{aligned} F_j &:= \mathcal{F}_{\mathrm{ser}}[f](j) = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\mathrm{e}^{-\mathrm{i}j\omega t}\,\mathrm{d}t, \\ G_j &:= \mathcal{F}_{\mathrm{ser}}[g](j), \\ A_j &:= \mathcal{F}_{\mathrm{ser}}[a](j), \\ B_j &:= \mathcal{F}_{\mathrm{ser}}[b](j), \\ q_j(\lambda) &:= \mathcal{F}_{\mathrm{ser}}[\hat{q}(\lambda;\cdot)](j), \end{aligned}$$

where $\omega := \frac{2\pi}{T}$. Then, by corollary 2.5.7,

$$f(t) = \sum_{j\in\mathbb{Z}} F_j \mathrm{e}^{\mathrm{i}j\omega t},$$

we have similar expressions for $g(t)$, $a(t)$, and $b(t)$, and

$$\hat{q}(\lambda;t) = \sum_{j\in\mathbb{Z}} q_j(\lambda)\mathrm{e}^{\mathrm{i}j\omega t} \implies \frac{\partial}{\partial t}\hat{q}(\lambda;t) = \sum_{j\in\mathbb{Z}} \mathrm{i}j\omega q_j(\lambda)\mathrm{e}^{\mathrm{i}j\omega t}.$$

Substituting these into 1, we find

$$\sum_{j\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}j\omega t}\left(\mathrm{i}\omega j + K\lambda^2\right) q_j(\lambda) = \sum_{j\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}j\omega t}\left(\mathrm{i}\lambda F_j + A_j - \mathrm{e}^{-\mathrm{i}\lambda}\mathrm{i}\lambda G_j - \mathrm{e}^{-\mathrm{i}\lambda}B_j\right).$$

Hence, by corollary 2.5.5 of the lecture notes, for all $j \in \mathbb{Z}$,

$$\left(\mathrm{i}\omega j + K\lambda^2\right) q_j(\lambda) = -K\left(\mathrm{i}\lambda\left(F_j - \mathrm{e}^{-\mathrm{i}\lambda}G_j\right) + A_j - \mathrm{e}^{-\mathrm{i}\lambda}B_j\right). \tag{2}$$

So we have reduced the original problem to that of finding the sequences $(A_j)_{j\in\mathbb{Z}}$ and $(B_j)_{j\in\mathbb{Z}}$.

Equation 2 holds for all λ such that $\hat{q}(\lambda;t)$ makes sense. But

$$\hat{q}(\lambda;t) = \int_{-\infty}^{\infty} q(x,t)\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}x = \int_0^1 q(x,t)\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}x.$$

The integrand is continuous, so we only have to check if the integral converges to something finite.

Suppose $\lambda = u + \mathrm{i}v$ for some $u, v \in \mathbb{R}$. Then

$$\begin{aligned} |\hat{q}(\lambda;t)| &= \left|\int_0^1 q(x,t)\mathrm{e}^{-\mathrm{i}ux}\mathrm{e}^{vx}\,\mathrm{d}x\right| \\ &\leqslant \int_0^1 |q(x,t)|\left|\mathrm{e}^{-\mathrm{i}ux}\right|\left|\mathrm{e}^{vx}\right|\,\mathrm{d}x \\ &= \int_0^1 |q(x,t)|\left|\mathrm{e}^{vx}\right|\,\mathrm{d}x \\ &\leqslant \mathrm{e}^{v}\max_{x\in[0,1]}|q(x,t)|\,(1-0) \\ &< \infty, \end{aligned}$$

because q is continuous. So equation 2 is true for all $\lambda \in \mathbb{C}$. In particular, it is true for those λ for which $\mathrm{i}\omega j + K\lambda^2 = 0$. For each $j \in \mathbb{Z} \setminus \{0\}$, there are two such λ, λ_j and $-\lambda_j$, such that

$$\lambda_j = \mathrm{e}^{-\operatorname{sgn}(j)\mathrm{i}\pi/4}\sqrt{\frac{\omega\,|j|}{K}}.$$

Note that this works for $j < 0$ as well as $j > 0$, since we are taking the square root of a positive number, and that because the point $\mathrm{e}^{\pm\mathrm{i}\pi/2}$ lies on the unit circle, we can take its square root by dividing the argument by 2.

We also check if $q_j(\pm\lambda_j)$ is finite:

$$\begin{aligned}
|q_j(\lambda)| &= \left|\frac{1}{T}\int_{-T/2}^{T/2} \hat{q}(\lambda;t)\mathrm{e}^{-\mathrm{i}j\omega t}\,\mathrm{d}t\right| \\
&\leqslant \frac{1}{T}\int_{-T/2}^{T/2} |\hat{q}(\lambda;t)|\left|\mathrm{e}^{-\mathrm{i}j\omega t}\right|\,\mathrm{d}t \\
&\leqslant \frac{1}{T}\left(\frac{T}{2} - \left(-\frac{T}{2}\right)\right)\max_{t\in[-T/2,T/2]} |\hat{q}(\lambda;t)| \\
&\leqslant \mathrm{e}^{\operatorname{Im}(\lambda)} \max_{t\in[-T/2,T/2],\, x\in[0,1]} |q(x,t)| \\
&< \infty,
\end{aligned}$$

because q is continuous in both x and t. So $q_j(\pm\lambda_j)$ is finite and we can say that the left hand side of equation 2 is zero at $\lambda = \pm\lambda_j$ for all $j \in \mathbb{Z} \setminus \{0\}$.

This yields two linear equations:

$$\begin{pmatrix} 1 & -\mathrm{e}^{-\mathrm{i}\lambda_j} \\ 1 & -\mathrm{e}^{\mathrm{i}\lambda_j} \end{pmatrix}\begin{pmatrix} A_j \\ B_j \end{pmatrix} = \begin{pmatrix} -\mathrm{i}\lambda_j\left(F_j - \mathrm{e}^{-\mathrm{i}\lambda_j}G_j\right) \\ \mathrm{i}\lambda_j\left(F_j - \mathrm{e}^{\mathrm{i}\lambda_j}G_j\right) \end{pmatrix}.$$

Since the columns $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -\mathrm{e}^{-\mathrm{i}\lambda_j} \\ -\mathrm{e}^{\mathrm{i}\lambda_j} \end{pmatrix}$ are linearly independent, the system has full rank and may be solved for A_j and B_j. Let $m = \mathrm{i}\lambda_j$. Then

$$\begin{aligned}
& \left(\begin{array}{cc|l} 1 & -\mathrm{e}^{-m} & -m\left(F_j - \mathrm{e}^{-m}G_j\right) \\ 1 & -\mathrm{e}^{m} & m\left(F_j - \mathrm{e}^{m}G_j\right) \end{array}\right) \\
\implies & \left(\begin{array}{cc|l} 1 & -\mathrm{e}^{-m} & m\left(F_j - \mathrm{e}^{-m}G_j\right) \\ 0 & \mathrm{e}^{m} - \mathrm{e}^{-m} & -2mF_j + \left(m\mathrm{e}^{-m} + m\mathrm{e}^{m}\right)G_j \end{array}\right) \\
\implies & \left(\begin{array}{cc|l} 1 & -\mathrm{e}^{-m} & m\left(F_j - \mathrm{e}^{-m}G_j\right) \\ 0 & 1 & \frac{-2m}{\mathrm{e}^{m}-\mathrm{e}^{-m}}F_j + \frac{m\left(\mathrm{e}^{-m}+\mathrm{e}^{m}\right)}{\mathrm{e}^{m}-\mathrm{e}^{-m}}G_j \end{array}\right),
\end{aligned}$$

noting that in the last line $m \neq 0$ so we are not dividing by 0. So for all $j \in \mathbb{Z} \setminus \{0\}$, we have

$$\begin{aligned}
B_j &= \frac{-2m}{\mathrm{e}^{m} - \mathrm{e}^{-m}}F_j + \frac{m\left(\mathrm{e}^{-m} + \mathrm{e}^{m}\right)}{\mathrm{e}^{m} - \mathrm{e}^{-m}}G_j \\
&= -m\operatorname{csch}(m)F_j + m\coth(m)G_j, \\
A_j &= \mathrm{e}^{-m}B_j + m\left(F_j - \mathrm{e}^{-m}G_j\right) \\
&= m(1 - \mathrm{e}^{-m}\operatorname{csch}(m))F_j + m\mathrm{e}^{-m}(\coth(m) - 1)G_j.
\end{aligned}$$

In the case where $j = 0$, equation 2 simplifies to

$$-\lambda^2 q_0(\lambda) = \mathrm{i}\lambda\left(F_0 - \mathrm{e}^{-\mathrm{i}\lambda}G_0\right) + A_0 - \mathrm{e}^{-\mathrm{i}\lambda}B_0.$$

Clearly $\lambda_0 = 0$, but that only gives one equation for two unknowns A_0 and B_0. To get another equation, differentiate equation 2 with respect to λ:

$$-\lambda^2 q_0'(\lambda) - 2\lambda q_0(\lambda) = \mathrm{i}F_0 + \lambda \mathrm{e}^{-\mathrm{i}\lambda} G_0 + \mathrm{i}\mathrm{e}^{-\mathrm{i}\lambda} B_0$$

We know that $q_0(\lambda)$ is finite. Noting that

$$\frac{\partial}{\partial\lambda}\hat{q}(\lambda;t) = \frac{\partial}{\partial\lambda}\int_0^1 q(x,t)\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}x = \int_0^1 (-\mathrm{i}x) q(x,t)\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}x,$$

we have

$$\begin{aligned}
\left|q_0'(\lambda)\right| &= \left|\frac{\partial}{\partial\lambda}\frac{1}{T}\int_{-T/2}^{T/2}\int_0^1 q(x,t)\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}x\,\mathrm{d}t\right| \\
&= \left|\frac{1}{T}\right|\left|\int_{-T/2}^{T/2}\int_0^1 (-\mathrm{i}x)q(x,t)\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\leqslant \frac{1}{T}\left|-\mathrm{i}\right|\int_{-T/2}^{T/2}\int_0^1 \left|xq(x,t)\right|\left|\mathrm{e}^{-\mathrm{i}\lambda x}\right|\,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant \frac{(T/2+T/2)}{T}\max_{t\in[-T/2,T/2]}\int_0^1 \left|xq(x,t)\right|\,\mathrm{d}x \\
&\leqslant \max_{t\in[-T/2,T/2],\,x\in[0,1]} \left|xq(x,t)\right|(1-0) \\
&< \infty,
\end{aligned}$$

since q is continuous in both x and t. So q_0' is also finite, and at $\lambda = 0$, the left side of the equation evaluates to 0. Together with equation 2 evaluated at 0, we get the linear system

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} 0 \\ -F_0 \end{pmatrix}.$$

This system has full rank (because it has 1s on the diagonal and is upper triangular, so the determinant is 1). So it can be solved for A_0 and B_0, and in this case $A_0 = B_0 = -F_0$.

But now we have $(A_j)_{j\in\mathbb{Z}}$ and $(B_j)_{j\in\mathbb{Z}}$. From these, we can reconstruct functions $a(t) = q_x(0,t)$ and $b(t) = q_x(1,t)$, using the formulae

$$a(t) = \sum_{j\in\mathbb{Z}} A_j \mathrm{e}^{\mathrm{i}j\omega t} \qquad \text{and} \qquad b(t) = \sum_{j\in\mathbb{Z}} B_j \mathrm{e}^{\mathrm{i}j\omega t}.$$

This completes the D to N map.

1.4 A generalisation from section 1.3

In section 1.3, the transform used to obtain the Q equation was the Fourier exponential transform. However, we may also use a transform with the basis extended by one dimension, to obtain a larger solution space for the same IBVP. Define

$$\mathcal{F}_\alpha[\varphi](\lambda) := \int_{-\infty}^{\infty} \varphi(x)\left(\alpha\mathrm{e}^{-\mathrm{i}\lambda x} + \beta\mathrm{e}^{\mathrm{i}\lambda x}\right)\,\mathrm{d}x$$

and

$$\hat{q}(\lambda;t) := \mathcal{F}_\alpha[q](\lambda;t) = \int_0^1 q(x,t)\left(\alpha\mathrm{e}^{-\mathrm{i}\lambda x} + \beta\mathrm{e}^{\mathrm{i}\lambda x}\right)\,\mathrm{d}x.$$

Note that

$$\begin{aligned}\mathcal{F}_\alpha[q_{xx}(x,t)](\lambda) &= \int_0^1 q_{xx}(x,t)\left(\alpha e^{-i\lambda x}+\beta e^{i\lambda x}\right) dx \\ &= \left[q_x(x,t)\left(\alpha e^{-i\lambda x}+\beta e^{i\lambda x}\right)\right]_{x=0}^{x=1} - \alpha(-i\lambda)\int_0^1 q_x(x,t)e^{-i\lambda x}\, dx - \beta(i\lambda)\int_0^1 q_x(x,t)e^{i\lambda x}\, dx \\ &= \left(\alpha e^{-i\lambda}+\beta e^{i\lambda}\right) q_x(1,t) - (\alpha+\beta)q_x(0,t) + \alpha i\lambda\left(q(1,t)e^{-i\lambda} - q(0,t)\right) \\ &\quad - \beta i\lambda\left(q(1,t)e^{i\lambda} - q(0,t)\right) - \lambda^2\int_0^1 q(x,t)\left(\alpha e^{-i\lambda x}+\beta e^{i\lambda x}\right) dx \\ &= \left(\alpha e^{-i\lambda}+\beta e^{i\lambda}\right) b(t) - (\alpha+\beta)a(t) + i\lambda\left(\alpha e^{-i\lambda}-\beta e^{i\lambda}\right) g(t) \\ &\quad - i\lambda(\alpha-\beta)f(t) - \lambda^2\hat{q}(\lambda;t).\end{aligned}$$

So applying $\mathcal{F}_\alpha$ to the heat equation gives us the Q equation

$$\begin{aligned}\frac{1}{K}\left[\frac{\partial}{\partial t}+K\lambda^2\right]\hat{q}(\lambda;t) &= \left(\alpha e^{-i\lambda}+\beta e^{i\lambda}\right) b(t) - (\alpha+\beta)a(t) \\ &\quad + i\lambda\left(\alpha e^{-i\lambda}-\beta e^{i\lambda}\right) g(t) - i\lambda(\alpha-\beta)f(t),\end{aligned}$$

which implies that

$$\begin{aligned}\frac{1}{K}\left(i\omega j+K\lambda^2\right) q_j(\lambda) &= \left(\alpha e^{-i\lambda}+\beta e^{i\lambda}\right) B_j - (\alpha+\beta)A_j \\ &\quad + i\lambda\left(\alpha e^{-i\lambda}-\beta e^{i\lambda}\right) G_j - i\lambda(\alpha-\beta)F_j.\end{aligned}$$

Note that setting $\alpha = 1, \beta = 0$ gives us equations 1 and 2. Then, picking the value of λ such that $i\omega j + K\lambda^2 = 0$, we have

$$\lambda_j = e^{i\pi/4}\sqrt{\frac{\omega j}{K}}$$

for positive integer j. So for $j \neq 0$, we have the system of linear equations

$$\begin{pmatrix} 1 & -e^{-i\lambda_j} \\ 1 & -e^{i\lambda_j}\end{pmatrix}\begin{pmatrix} A_j \\ B_j\end{pmatrix} = \begin{pmatrix} -i\lambda_j\left(F_j - e^{-i\lambda_j}G_j\right) \\ i\lambda_j\left(F_j - e^{i\lambda_j}G_j\right)\end{pmatrix}$$

as before, with the first row corresponding to $\alpha = 1, \beta = 0$, and the second corresponding to $\alpha = 0, \beta = 1$. For $j = 0$, setting $\alpha = 0, \beta = 1$ gives us the same equation as before. So it is still necessary to differentiate equation 2 with respect to λ to obtain another equation.

1.5 The Laplacian operator

The Laplacian differential operator ∇^2 in Cartesian spatial coordinates (x, y, z) and temporal variable t is given by

$$\nabla^2 u(x,y,z;t) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) u(x,y,z;t).$$

The Laplacian is defined as the divergence of the gradient of a function.

It would be useful to express the Laplacian in some other coordinate systems we will be working with. First, in domains with some circular symmetry, polar cylindrical coordinates are convenient to use. Cartesian coordinates are expressed in terms of these as follows:

$$\begin{aligned} x &= \rho\cos\varphi, \\ y &= \rho\sin\varphi, \\ z &= z,\end{aligned}$$

where ρ is the radial distance from a point to the z-axis and φ is the angle measured from the positive x-axis to the radial projection onto the xy-plane. Since the z-coordinate remains unchanged, for the purpose of deriving the Laplacian in cylindrical systems, we can ignore the z-coordinate and work in two dimensions. Using $\vec{i}$ and $\vec{j}$ as the standard unit vectors in the positive x and y directions, the position vector can be written as

$$\vec{r} = x\vec{i} + y\vec{j} = -\sin\varphi\vec{i} + \cos\varphi\vec{j}.$$

Then, the unit basis vectors in polar coordinates are

$$\begin{aligned}
\vec{e_\varphi} &= \frac{-\rho\sin\varphi\vec{i} + \rho\cos\varphi\vec{j}}{\sqrt{(\rho\sin\varphi)^2 + (\rho\cos\varphi)^2}} \\
&= -\sin\varphi\vec{i} + \cos\varphi\vec{j}, \\
\vec{e_\rho} &= \frac{\cos\varphi\vec{i} + \sin\varphi\vec{j}}{\sqrt{\cos^2\varphi + \sin^2\varphi}} \\
&= \cos\varphi\vec{i} + \sin\varphi\vec{j}.
\end{aligned}$$

So we have

$$\begin{aligned}
\vec{i} &= \cos\varphi\vec{e_\rho} - \sin\varphi\vec{e_\varphi}, \\
\vec{j} &= \sin\varphi\vec{e_\rho} + \cos\varphi\vec{e_\varphi}.
\end{aligned}$$

We can also find the partial derivatives of the polar unit basis vectors with respect to polar coordinates to be used later:

$$\begin{aligned}
\frac{\partial\vec{e_\rho}}{\partial\rho} &= 0, \\
\frac{\partial\vec{e_\rho}}{\partial\varphi} &= -\sin\varphi\vec{i} + \cos\varphi\vec{j} = \vec{e_\varphi}, \\
\frac{\partial\vec{e_\varphi}}{\partial\rho} &= 0, \\
\frac{\partial\vec{e_\varphi}}{\partial\varphi} &= -\cos\varphi\vec{i} - \sin\varphi\vec{j} = -\vec{e_\rho}.
\end{aligned}$$

Using the chain rule, we can express the Cartesian partial derivatives in terms of polar coordinates:

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial x} \\
&= \frac{\partial}{\partial \rho}\frac{2x}{2\sqrt{x^2+y^2}} - \frac{\partial}{\partial \varphi}\frac{y}{x^2+y^2} \\
&= \frac{\partial}{\partial \rho}\frac{\rho\cos\varphi}{\rho} - \frac{\partial}{\partial \varphi}\frac{\rho\sin\varphi}{\rho^2} \\
&= \frac{\partial}{\partial \rho}\cos\varphi - \frac{\partial}{\partial \varphi}\frac{\sin\varphi}{\rho}, \\
\frac{\partial}{\partial y} &= \frac{\partial}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial y} \\
&= \frac{\partial}{\partial \rho}\sin\varphi - \frac{\partial}{\partial \varphi}\frac{x}{x^2+y^2} \\
&= \frac{\partial}{\partial \rho}\sin\varphi - \frac{\partial}{\partial \varphi}\frac{\cos\varphi}{\rho}.
\end{aligned}$$

Then, we can express the gradient in polar coordinates.

$$\begin{aligned}
\nabla &= \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} \\
&= \left(\frac{\partial}{\partial \rho}\cos\varphi - \frac{\partial}{\partial \varphi}\frac{\sin\varphi}{\rho}\right)(\cos\varphi\vec{e_\rho} - \sin\varphi\vec{e_\varphi}) + \left(\frac{\partial}{\partial \rho}\sin\varphi + \frac{\partial}{\partial \varphi}\frac{\cos\varphi}{\rho}\right)(\sin\varphi\vec{e_\rho} + \cos\varphi\vec{e_\varphi}) \\
&= \left(\frac{\partial}{\partial \rho}\cos^2\varphi + \frac{\partial}{\partial \rho}\sin^2\varphi - \frac{\partial}{\partial \varphi}\frac{\sin\varphi\cos\varphi}{\rho} + \frac{\partial}{\partial \varphi}\frac{\sin\varphi\cos\varphi}{\rho}\right)\vec{e_\rho} \\
&\quad + \left(-\frac{\partial}{\partial \rho}\cos\varphi\sin\varphi + \frac{\partial}{\partial \rho}\cos\varphi\sin\varphi + \frac{\partial}{\partial \varphi}\frac{\sin^2\varphi}{\rho} + \frac{\partial}{\partial \varphi}\frac{\cos^2\varphi}{\rho}\right)\vec{e_\varphi} \\
&= \frac{\partial}{\partial \rho}\vec{e_\rho} + \frac{1}{\rho}\frac{\partial}{\partial \varphi}\vec{e_\varphi}.
\end{aligned}$$

We also express divergence in polar coordinates:

$$\begin{aligned}
\nabla\cdot &= \left(\frac{\partial}{\partial \rho}\vec{e_\rho} + \frac{1}{\rho}\vec{e_\varphi}\right)\cdot(F_\rho\vec{e_\rho} + F_\varphi\vec{e_\varphi}) \\
&= \vec{e_\rho}\left(\frac{\partial F_\rho}{\partial \rho}\vec{e_\rho} + F_\rho\frac{\partial \vec{e_\rho}}{\partial \rho} + \frac{\partial F_\varphi}{\partial \rho}\vec{e_\varphi} + F_\varphi\frac{\partial \vec{e_\varphi}}{\partial \rho}\right) \\
&\quad + \frac{1}{\rho}\vec{e_\varphi}\left(\frac{\partial F_\rho}{\partial \varphi}\vec{e_\rho} + F_\rho\frac{\partial \vec{e_\rho}}{\partial \varphi} + \frac{\partial F_\varphi}{\partial \varphi}\vec{e_\varphi} + F_\varphi\frac{\partial \vec{e_\varphi}}{\partial \varphi}\right) \\
&= \frac{\partial F_\rho}{\partial \rho} + \frac{1}{\rho}F_\rho + \frac{1}{\rho}\frac{\partial F_\varphi}{\partial \varphi},
\end{aligned}$$

using earlier results and orthogonality of the polar basis vectors. Finally, the Laplacian in polar coordinates is

$$\begin{aligned}
\nabla^2 &= \nabla\cdot\nabla \\
&= \left(\frac{\partial F_\rho}{\partial \rho} + \frac{1}{\rho}F_\rho + \frac{1}{\rho}\frac{\partial F_\varphi}{\partial \varphi}\right)\cdot\left(\frac{\partial}{\partial \rho}\vec{e_\rho} + \frac{1}{\rho}\frac{\partial}{\partial \varphi}\vec{e_\varphi}\right) \\
&= \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}.
\end{aligned}$$

In polar cylindrical coordinates, the Laplacian is then

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}. \tag{3}$$

Now, in spherical coordinates, we have

$$\begin{aligned} x &= r\cos\varphi\sin\theta, \\ y &= r\sin\varphi\sin\theta, \\ z &= r\cos\theta, \end{aligned}$$

where r is the radial distance from the origin to the point (x, y, z), the angle φ is defined above, and θ is the angle from the positive z-axis. More conveniently, we can express spherical coordinates in terms of polar cylindrical coordinates,

$$\begin{aligned} r &= \sqrt{\rho^2 + z^2}, \\ \theta &= \arctan\left(\frac{\rho}{z}\right), \\ \varphi &= \varphi, \end{aligned}$$

and conversely,

$$\rho = r\sin\theta,$$

which gives us

$$\begin{aligned} x &= \rho\cos\varphi, \\ y &= \rho\sin\varphi. \end{aligned}$$

Notice that (z, ρ) are obtained from (r, θ) in the same way that (x, y) are obtained from (ρ, φ). So by the preceding argument, we have

$$\frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Adding this to the polar Laplacian in two dimensions, we get

$$\begin{aligned} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial z^2} &= \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \\ \implies \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} &= \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \end{aligned}$$

So it remains to compute $\frac{\partial}{\partial \rho}$. By the chain rule,

$$\begin{aligned} \frac{\partial}{\partial \rho} &= \frac{\partial}{\partial r}\frac{\partial r}{\partial \rho} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial \rho} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial \rho} \\ &= \frac{\partial}{\partial r}\frac{\rho}{\sqrt{\rho^2 + z^2}} + \frac{\partial}{\partial \theta}\frac{1/z}{1 + \rho^2/z^2} + 0 \\ &= \frac{\partial}{\partial r}\frac{\rho}{r} + \frac{\partial}{\partial \theta}\frac{\cos\theta}{r}. \end{aligned}$$

Substituting this and $\rho = r\sin\theta$ into the above, we obtain the Laplacian in spherical coordinates,

$$\nabla^2 = \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{4}$$

We now turn to an entirely different coordinate system, that of parabolic cylindrical coordinates. The coordinates (σ, τ, z) are given in terms of Cartesian coordinates by

$$\begin{aligned} x &= \sigma\tau, \\ y &= \frac{1}{2}(\tau^2 - \sigma^2), \\ z &= z. \end{aligned}$$

Here, the coordinate surfaces are confocal parabolic cylinders, with constant σ surfaces opening towards the positive y-axis, and constant τ surfaces opening towards the negative y-axis. As with polar cylindrical coordinates, we can ignore the z-coordinate for now. We first compute

$$\begin{aligned} \frac{\partial \sigma}{\partial x} &= \frac{1}{\tau}, \\ \frac{\partial \sigma}{\partial y} &= \frac{-1}{\sqrt{\tau^2 - 2y}} = \frac{-1}{\sigma}, \\ \frac{\partial \tau}{\partial x} &= \frac{1}{\sigma}, \\ \frac{\partial \tau}{\partial y} &= \frac{1}{\sqrt{(\tau^2 - \sigma^2) + \sigma^2}} = \frac{1}{\tau}. \end{aligned}$$

By the chain rule,

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial \sigma}\frac{\partial \sigma}{\partial x} + \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial x}, \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial \sigma}\frac{\partial \sigma}{\partial y} + \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial y}. \end{aligned}$$

Then making the appropriate substitutions, we obtain

$$\begin{aligned} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} &= \left(\frac{1}{\tau}\frac{\partial}{\partial \sigma} + \frac{1}{\sigma}\frac{\partial}{\partial \tau}\right)^2 + \left(\frac{-1}{\sigma}\frac{\partial}{\partial \sigma} + \frac{1}{\tau}\frac{\partial}{\partial \tau}\right)^2 \\ &= \frac{1}{\tau^2}\frac{\partial^2}{\partial \sigma^2} + \frac{1}{\tau\sigma}\left(\frac{\partial^2}{\partial \sigma \partial \tau} + \frac{\partial^2}{\partial \tau \partial \sigma}\right) + \frac{1}{\sigma^2}\frac{\partial^2}{\partial \tau^2} + \frac{1}{\sigma^2}\frac{\partial^2}{\partial \sigma^2} - \frac{1}{\sigma\tau}\left(\frac{\partial^2}{\partial \tau \partial \sigma} + \frac{\partial^2}{\partial \sigma \partial \tau}\right) + \frac{1}{\tau^2}\frac{\partial^2}{\partial \tau^2} \\ &= \frac{1}{\tau^2 + \sigma^2}\left(\frac{\partial^2}{\partial \sigma^2} + \frac{\partial^2}{\partial \tau^2}\right), \end{aligned}$$

where we have assumed that the necessary conditions for Clairaut's theorem (equality of mixed partials) hold to justify cancelling terms in the last line. So the Laplacian in parabolic cylindrical coordinates is

$$\nabla^2 = \frac{1}{\tau^2 + \sigma^2}\left(\frac{\partial^2}{\partial \sigma^2} + \frac{\partial^2}{\partial \tau^2}\right) + \frac{\partial^2}{\partial z^2}. \tag{5}$$

1.6 Separation of variables

In the following sections, we will use the technique of separation of variables to express partial differential equations (PDEs) as ordinary differential equations (ODEs), as ODEs are generally easier to solve. This technique involves looking for solutions in the form $u(x, y) = X(x)Y(y)$, and then obtaining ODEs for $X(x)$ and $Y(y)$. The ODEs will contain a *separation constant.* The function $u(x, y)$ is called a *separated solution.*

To illustrate this technique, consider Laplace's equation in two dimensions in rectangular coordinates,

$$u_{xx} + u_{yy} = 0.$$

We begin by assuming that the solution is in the form $u(x, y) = X(x)Y(y)$. Then we have

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Dividing by u, we get

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0.$$

Each term depends on only one of the variables. This equation is only true if both terms are constants that sum to 0, so we introduce a separation constant λ and obtain the ODEs

$$X''(x) - \lambda X(x) = 0,$$
$$Y''(y) + \lambda Y(y) = 0.$$

In general, $X(x)$ and $Y(y)$ may be real-valued or complex-valued. If they are complex-valued, then the separation constant is also complex. If $u(x, y)$ is the solution to an inhomogeneous linear PDE, then $\operatorname{Re} u(x, y)$ satisfies the same PDE, and $\operatorname{Im} u(x, y)$ satisfies the corresponding homogeneous PDE. Further, if a linear PDE is constant coefficient, the solutions can always be found and may be written in the form $u = \mathrm{e}^{\alpha x}\mathrm{e}^{\beta y}$, where $\alpha, \beta \in \mathbb{C}$. If the PDE is not constant coefficient, however, it is not guaranteed that the equation will have any nonconstant separated solution. Nonetheless, certain classes of equations can still be solved by separation of variables, such as equations of the form

$$a(x)u_{xx} + c(y)u_{yy} + d(x)u_x + e(y)u_y = 0.$$

We may also impose additional conditions on the separated solutions, such as boundary conditions (BCs) or boundedness conditions. BCs specify how the solution and/or its derivatives behave at the boundary of the (spatial) domain. (A BC on the temporal variable, at $t = 0$, is usually referred to instead as an initial condition (IC).) Boundedness conditions, specified for values of t on the whole real line, describe the behaviour of systems over a long period of time.

1.6.1 Application to heat equation

We now examine separation of variables for the heat equation in non-rectangular coordinate systems, in one temporal and three spatial dimensions. A solution u to the heat equation in general is

$$u_t = K\nabla^2 u.$$

In polar cylindrical coordinates, the heat equation becomes

$$u_t - K\left(u_{\rho\rho} + \frac{1}{\rho}u_\rho + \frac{1}{\rho^2}u_{\varphi\varphi} + u_{zz}\right) = 0.$$

We want to find, by repeated separation of variables, solutions in the form

$$u(\rho, \varphi, z; t) = R(\rho)\Phi(\varphi)Z(z)T(t).$$

Substituting this into the heat equation and dividing by Ku, we get

$$\frac{T'}{KT} - \left(\frac{R'' + R'/\rho}{R} + \frac{\Phi''/\rho^2}{\Phi} + \frac{Z''}{Z}\right) = 0.$$

Let $\lambda = \frac{T'}{KT}$, so we get

$$T' - \lambda KT = 0,$$
$$\frac{R'' + R'/\rho}{R} + \frac{\Phi''/\rho^2}{\Phi} + \frac{Z''}{Z} = \lambda.$$

Let $-\mu = \frac{Z''}{Z}$, then

$$\begin{aligned} Z'' + \mu Z &= 0, \\ \frac{R'' + R'/\rho}{R} + \left(\frac{1}{\rho^2}\right)\frac{\Phi''}{\Phi} &= \lambda + \mu. \end{aligned}$$

Let $-\nu = \frac{\Phi''}{\Phi}$, then

$$\begin{aligned} \Phi'' + \nu\Phi &= 0, \\ \frac{R'' + R'/\rho}{R} &= \lambda + \mu + \frac{\nu}{\rho^2} \\ \implies R'' + \left(\frac{1}{\rho}\right)R' - \left(\lambda + \mu + \frac{\nu}{\rho^2}\right)R &= 0. \end{aligned}$$

So we have obtained one ODE for each of the four variables, as well as three separation constants λ, μ, ν.

Now in spherical coordinates, the heat equation is

$$u_t - K\left(\frac{2}{r}u_r + u_{rr} + \frac{\cos\theta}{\sin^2\theta}u_\theta + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r^2\sin^2\theta}u_{\varphi\varphi}\right) = 0.$$

We are looking for solutions in the form

$$u(r,\theta,\varphi;t) = R(r)\Theta(\theta)\Phi(\varphi)T(t).$$

Substituting into the heat equation and dividing by Ku, we get

$$\frac{T'}{KT} - \left(\frac{2R'/r + R''}{R} + \frac{(\cot\theta)\Theta' + \Theta''}{r^2\Theta} + \frac{\Phi''}{r^2(\sin^2\theta)\Phi}\right) = 0.$$

By a similar process as above, we let

$$\begin{aligned} -\lambda &= \frac{T'}{KT}, \\ -\mu &= r^2\left(\frac{2R'/r + R''}{R} + \lambda\right), \\ -\nu &= \frac{\Phi''}{\Phi}. \end{aligned}$$

Then,

$$\begin{aligned} T' + \lambda KT &= 0, \\ r^2\left(\frac{2}{r}R' + R'' + \lambda R\right) + \mu R &= 0, \\ \Phi'' + \nu\Phi &= 0, \\ r^2\left(\frac{R'/r + R''}{R}\right) + \frac{(\cot\theta)\,\Theta' + \Theta''}{\Theta} + \frac{\Phi''}{(\sin^2\theta)\Phi} &= -\lambda r^2 \\ \implies \sin^2\theta\left((\cot\theta)\,\Theta' + \Theta'' - \mu\Theta\right) - \nu\Theta &= 0. \end{aligned}$$

Once again, we obtain four ODEs and three separation constants.

2 D to N map for cylindrical heat problems

2.1 Heat equation on a disc

We now turn to solving the D to N map using the Q equation method for various problems in polar cylindrical coordinates, beginning with the simplest case of a disc. Suppose we want to find a solution to an IBVP in the form

$$u(\rho,\varphi,t) = q(\rho,t)\Phi(\varphi).$$

Define a separation constant $-\mu = \Phi''/\Phi$, which gives us two equations

$$\Phi'' + \mu\Phi = 0,$$
$$q_t = q_{\rho\rho} + \frac{1}{\rho}q_\rho - \frac{\mu}{\rho^2}.$$

The first of these equations is an ODE which can be solved with $\Phi(-\pi) = \Phi(\pi)$ and $\Phi'(-\pi) = \Phi'(\pi)$, which gives us

$$\Phi(\varphi) = A\cos(m\varphi) + B\sin(m\varphi),$$

where $m \in \mathbb{N}_0$ and $\mu = m^2$. Let

$$\mathcal{H}_m[f](k) = \int_0^a \rho f(\rho) J_m(k\rho)\,\mathrm{d}\rho,$$

where $k \in \mathbb{C}$ and J_m is the mth cylindrical Bessel function of the first kind, which is entire in k for all nonnegative integers m. Then

$$\mathcal{H}_m[q_t(\cdot,t)](k) = \mathcal{H}_m\left[q_{\rho\rho}(\cdot,t) + \frac{1}{\rho}q_\rho(\cdot,t) - \frac{\mu^2}{\rho^2}\right](k).$$

Consider the derivative of $\mathcal{H}_m[R]$ with respect to t. Defining a differential operator $L_m R := R''(\rho) + \frac{1}{\rho}R'(\rho) - \frac{\mu^2}{\rho^2}R(\rho)$, we have

$$\begin{aligned}
\mathcal{H}_m[L_m R](k) &= \int_0^a \rho L_m R(\rho) J_m(k\rho)\,\mathrm{d}\rho \\
&= \int_0^a \left[\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho R'(\rho)\right) - \frac{\mu^2}{\rho}\right] J_m(k\rho)\,\mathrm{d}\rho \\
&= \left[\rho R'(\rho) J_m(k\rho)\right]_{\rho=0}^{\rho=a} - k\int_0^a \rho R'(\rho) J_m'(k\rho)\,\mathrm{d}\rho - \int_0^a \frac{\mu^2}{\rho} J_m(k\rho) R(\rho)\,\mathrm{d}\rho \\
&= aR'(a)J_m(ka) - \left[R(\rho)(k\rho)J_m'(k\rho)\right]_{\rho=0}^{\rho=a} + k\int_0^a R(\rho)\frac{\mathrm{d}}{\mathrm{d}(k\rho)}\left(k\rho J_m'(k\rho)\right)\,\mathrm{d}\rho \\
&\quad - \int_0^a \frac{\mu^2}{\rho} J_m(k\rho) R(\rho)\,\mathrm{d}\rho \\
&= aR'(a)J_m(ka) - kaR(a)J_m'(ka) - k^2\int_0^a \rho R(\rho) J_m(k\rho)\,\mathrm{d}\rho \\
&\quad + m^2\int_0^a \frac{1}{\rho} J_m(k\rho) R(\rho)(1-1)\,\mathrm{d}\rho,
\end{aligned}$$

where we have used $\frac{\mathrm{d}}{\mathrm{d}(k\rho)}\left(k\rho J'm(k\rho)\right) = \left(-k\rho + \frac{m^2}{k\rho}\right)J_m(k\rho)$, because J_m satisfies Bessel's equation with $\mu = m^2$,

$$= -k^2\mathcal{H}_m[R](k) + aR'(a)J_m(ka) - kaR(a)J_m'(ka).$$

So
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{H}_m[q](k;t)\right) = -k^2\mathcal{H}_m[q](k;t) + aq_\rho(a,t)J_m(ka) - kaq(a,t)J_m'(ka).$$
Now define
$$Q(k,t) := -\mathcal{H}_m[q](k;t).$$
Then
$$Q_t + k^2Q = akq(a,t)J_m(ka) - aq_\rho(a,t)J_m(ka).$$
This is a Q equation relating $q(a,t)$ to $q_\rho(a,t)$. (There is no $q(0,t)$, because $\rho = 0$ is an interior point of the domain.) Therefore, under assumption of time periodicity of q, we have a D to N map that is asymptotically valid.

Consider the following problem. Suppose $q(\rho,t)$ satisfies the heat equation, an initial condition and a Dirichlet BC on the domain $\rho \in [0,a]$ and $t \in [0,\infty)$:

$$\begin{aligned} q_t &= K\nabla^2 q, \\ q(a,t) &= f(t), \\ q(\rho,0) &= P(x), \end{aligned}$$

where $f(t)$ is known and we seek $b(t) := q_\rho(a,t)$. Then the Q equation is

$$Q_t + k^2Q = aJ_m(ka)\left(kf(t) - b(t)\right).$$

Following a similar process as we did in subsection 1.3, we apply the Fourier exponential series transform on the time variable. Denote

$$\begin{aligned} F_j &:= \mathcal{F}_{\text{ser}}[f](j), \\ B_j &:= \mathcal{F}_{\text{ser}}[b](j), \\ q_j(k) &:= \mathcal{F}_{\text{ser}}[Q(k;\cdot)](j). \end{aligned}$$

Then applying the Fourier inverse transform (having extended the domain of t appropriately) we have

$$\begin{aligned} f(t) &= \sum_{j\in\mathbb{Z}} F_j \mathrm{e}^{\mathrm{i}j\omega t}, \\ b(t) &= \sum_{j\in\mathbb{Z}} B_j \mathrm{e}^{\mathrm{i}j\omega t}, \\ Q(k;t) &= \sum_{j\in\mathbb{Z}} q_j(k) \mathrm{e}^{\mathrm{i}j\omega t} \\ \implies Q_t &= \sum_{j\in\mathbb{Z}} \mathrm{i}j\omega q_j(k) \mathrm{e}^{\mathrm{i}j\omega t}. \end{aligned}$$

Substituting these into the Q equation, we get

$$\sum_{j\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}j\omega t}\left(\mathrm{i}j\omega + k^2\right)q_j(k) = \sum_{j\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}j\omega t}\left(kF_j - B_j\right)aJ_m(ka).$$

Hence, for all $j \in \mathbb{Z}$,
$$\left(\mathrm{i}j\omega + k^2\right)q_j(k) = \left(kF_j - B_j\right)aJ_m(ka). \tag{6}$$

We want to say that the left hand side of equation 6 is zero if $\mathrm{i}j\omega + k^2 = 0$. So we show that $Q(k,t)$ is finite:

$$\begin{aligned}
|-\mathcal{H}_m[q](k;t)| &= \left|\int_0^a \rho q(\rho,t) J_m(k\rho)\,\mathrm{d}\rho\right| \\
&\leqslant \int_0^a |\rho q(\rho,t) J_m(k\rho)|\,\mathrm{d}\rho \\
&\leqslant \max_{\rho\in[0,a]} |\rho q(\rho,t) J_m(k\rho)| \\
&< \infty,
\end{aligned}$$

since q is differentiable (hence continuous) in ρ, and J_m is entire. By an earlier argument in subsection 1.3 (replacing λ with k and $\hat{q}$ with Q), this implies that $q_j(k)$ is also finite. Then equation 6 holds for all $k \in \mathbb{C}$, and it holds in particular for k such that

$$\mathrm{i}j\omega + k^2 = 0 \implies k_j = \mathrm{e}^{-\operatorname{sgn}(j)\mathrm{i}\pi/4}\sqrt{\omega\,|j|}$$

for all $j \in \mathbb{Z}\setminus\{0\}$. So at $k = \pm k_j$, we have

$$(k_j F_j - B_j)\, a J_m(k_j a) = 0,$$

which implies that $B_j = k_j F_j$. When $j = 0$, equation 6 becomes

$$k^2 q_0(k) = (k_0 F_0 - B_0)\, a J_m(ka).$$

We know $k_0 = 0$, so $B_0 = 0$. This completes the D to N map, as we have found $(B_j)_{j\in\mathbb{Z}}$.

In the case where instead of the problem above, we know the Neumann BC $b(t)$ and wish to find $f(t)$, the argument is largely the same. For nonzero j, we have $F_j = \frac{B_j}{k_j}$, which is not a problem as k_j is nonzero. However, in the last step for $j = 0$, we cannot find a value for F_0, so we differentiate the coefficient equation with respect to k and obtain

$$k^2 q_0'(k) + 2k q_0(k) = a F_0 \left(J_m(ka) + k J_m'(ka)\right).$$

Then at $k = k_0$, assuming $q_0'(k)$ is finite, we have

$$a F_0 J_m(0) = 0 \implies F_0 = 0,$$

since this equation has to hold for all values of m. We check that $q_0'(k)$ is indeed finite:

$$\begin{aligned}
\left|q_0'(k)\right| &= \left|\frac{\partial}{\partial k}\mathcal{F}_{\mathrm{ser}}[Q(k;\cdot)](0)\right| \\
&= \left|\frac{\partial}{\partial k}\frac{1}{T}\int_{-T/2}^{T/2} Q(k;t)\mathrm{e}^0\,\mathrm{d}t\right| \\
&= \left|\frac{1}{T}\frac{\partial}{\partial k}\int_{-T/2}^{T/2}\int_0^a \rho q(\rho,t) J_m(k\rho)\,\mathrm{d}\rho\,\mathrm{d}t\right| \\
&= \left|\frac{1}{T}\right|\left|\int_{-T/2}^{T/2}\int_0^a \rho^2 q(\rho,t) J_m'(k\rho)\,\mathrm{d}\rho\,\mathrm{d}t\right| \\
&\leqslant \frac{1}{T}\max_{t\in[-T/2,T/2],\,\rho\in[0,a]} \left|\rho^2 q(\rho,t) J_m'(k\rho)\right| (T/2 + T/2)(a-0) \\
&< \infty,
\end{aligned}$$

since q is continuous in both ρ and t, and J_m' is entire.

Now consider the problem involving a Robin BC, where we wish to find both the Dirichlet and Neumann BCs. That is, on the domain $\rho \in [0, a]$ and $t \in [0, \infty)$, we have that $q(\rho, t)$ satisfies:

$$\begin{aligned} q_t &= K\nabla^2 q, \\ q(a,t) + \beta q_\rho(a,t) &= g(t), \\ q(\rho, 0) &= P(x), \end{aligned}$$

where $g(t)$ and constant $\beta \in \mathbb{R}$ are known, and we seek $f(t)$ and $b(t)$ (as previously defined). We denote $G_j := \mathcal{F}_{\text{ser}}[g](j)$ and get $g(t) = \sum_{j\in\mathbb{Z}} G_j e^{ij\omega t}$. Applying $\mathcal{F}_{\text{ser}}$ to the boundary condition, we have

$$F_j + \beta B_j = G_j.$$

Combining this with our earlier observation that for $j \neq 0$,

$$k_j F_j - B_j = 0,$$

we have two linear equations in two unknowns. By substitution, we get

$$\begin{aligned} F_j &= \frac{G_j}{1+\beta k_j}, \\ B_j &= k_j \left(\frac{G_j}{1+\beta k_j} \right). \end{aligned}$$

We are not dividing by zero here, because k_j cannot be purely real (or purely imaginary), so $\beta k_j \neq -1$. For $j = 0$, we still have

$$F_0 = B_0 = 0.$$

So we have found $(F_j)_{j\in\mathbb{Z}}$ and $(B_j)_{j\in\mathbb{Z}}$, which can be used to reconstruct $f(t)$ and $b(t)$, as desired.

2.2 Heat equation on an annulus

On an annulus, the centre of the disc is excluded from the domain, giving the spatial domain $\rho \in (a, b)$ where $a, b \in \mathbb{R}$. For this reason, we may introduce the mth cylindrical Bessel function of the second kind in the transform, Y_m. (We could not do this previously because Y_m has a singularity at 0.) Define

$$\mathcal{B}_m[f](k) := \int_a^b \rho f(\rho) \left(\alpha J_m(k\rho) + \beta Y_m(k\rho) \right),$$

where $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned} &\mathcal{B}_m[q_k(\cdot,t)](k) = \mathcal{B}\left[q_{\rho\rho}(\cdot,t) + \frac{1}{\rho} q_\rho(\cdot,t) - \frac{m^2}{\rho^2} \right](k) \\ \implies &\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{B}_m[q_k(\cdot,t)](k) = -k^2 \mathcal{B}_m[q](k;t) + b\left(\alpha J_m(kb) + \beta Y_m(kb) \right) q_\rho(b,t) - kb\left(\alpha J_m'(kb) + \beta Y_m'(kb) \right) q(b,t) \\ &\qquad - a\left(\alpha J_m(ka) + \beta Y_m(ka) \right) q_\rho(a,t) + ka\left(\alpha J_m'(ka) + \beta Y_m'(ka) \right) q_(a,t). \end{aligned}$$

Define

$$Q(k,t) := -\mathcal{B}_m[q](k;t).$$

Then we have the Q equation

$$k^2Q + Q_t = -ka\left(\alpha J_m'(ka) + \beta Y_m'(ka)\right) q(a,t) + kb\left(\alpha J_m'(kb) + \beta Y_m'(kb)\right) q(b,t) \\ + a\left(\alpha J_m(ka) + \beta Y_m(ka)\right) q_\rho(a,t) - b\left(\alpha J_m(kb) + \beta Y_m(kb)\right) q_\rho(b,t).$$

On an annulus, we have BCs at both a and b. Consider first an IBVP with no Robin BCs (i.e. only Dirichlet and Neumann BCs). Denote

$$\begin{aligned} q(a,t) &=: d(t), \\ q(b,t) &=: f(t), \\ q_\rho(a,t) &=: g(t), \\ q_\rho(b,t) &=: h(t). \end{aligned}$$

By a similar process of applying $\mathcal{F}_{\text{ser}}$ on t, the Q equation is reduced to

$$\left(\mathrm{i}j\omega + k^2\right) q_j(k) = -ka\left(\alpha J_m'(ka) + \beta Y_m'(ka)\right) D_j + kb\left(\alpha J_m'(kb) + \beta Y_m'(kb)\right) F_j \\ + a\left(\alpha J_m(ka) + \beta Y_m(ka)\right) G_j - b\left(\alpha J_m(kb) + \beta Y_m(kb)\right) H_j.$$

Again, for all $j \in \mathbb{Z} \setminus \{0\}$, we have

$$k_j = \mathrm{e}^{-\operatorname{sgn}(j)\mathrm{i}\pi/4}\sqrt{\omega\,|j|},$$

assuming $q_j(k)$ is finite. Since $\mathcal{B}_m[f](k)$ is a linear combination

$$\alpha\int_a^b \rho f(\rho) J_m(k\rho)\,\mathrm{d}\rho + \beta\int_a^b \rho f(\rho) Y_m(k\rho)\,\mathrm{d}\rho,$$

and J_m, Y_m are both entire on $\rho \in (a,b)$, we may use similar arguments as we did to show $\mathcal{H}_m[f](k)$ was finite in section 2.1. Then $q_j(k)$ is also finite, by a previous argument. So for $k = \pm k_j$,

$$0 = -k_ja\left(\alpha J_m'(k_ja) + \beta Y_m'(k_ja)\right) D_j + k_jb\left(\alpha J_m'(k_jb) + \beta Y_m'(k_jb)\right) F_j \\ + a\left(\alpha J_m(k_ja) + \beta Y_m(k_ja)\right) G_j - b\left(\alpha J_m(k_jb) + \beta Y_m(k_jb)\right) H_j.$$

To obtain two linear equations, we first set $\alpha = 1, \beta = 0$ to get

$$-k_jaJ_m'(k_ja)D_j + k_jbJ_m'(k_jb)F_j + aJ_m(k_ja)G_j - bJ_m(k_jb)H_j = 0. \tag{7}$$

We then set $\alpha = 0, \beta = 1$ to get

$$-k_jaY_m'(k_ja)D_j + k_jbY_m'(k_jb)F_j + aY_m(k_ja)G_j - bY_m(k_jb)H_j = 0. \tag{8}$$

Consider the case where the Dirichlet BCs $d(t), f(t)$ are known and the Neumann BCs $g(t), h(t)$ are unknown. Then the system of linear equations is

$$\begin{pmatrix} aJ_m(k_ja) & -bJ_m(k_jb) \\ aY_m(k_ja) & -bY_m(k_jb) \end{pmatrix}\begin{pmatrix} G_j \\ H_j \end{pmatrix} = \begin{pmatrix} -k_jaJ_m'(k_ja)D_j + k_jbJ_m'(k_jb)F_j \\ -k_jaY_m'(k_ja)D_j + k_jbY_m'(k_jb)F_j \end{pmatrix}.$$

By definition, $J_m(z)$ and $Y_m(z)$ are linearly independent. Note that for $j \neq 0$, k_j is never real. Since the zeros of J_m for $m \geqslant 0$ are all real, and $a, b \in \mathbb{R}$, it follows that $J_m(k_ja), J_m(k_jb)$ are never zero. Similarly, $Y_m(k_ja), Y_m(k_jb)$ are never zero. So this system has full rank.

For $j = 0$, we first divide equation 8 by $Y_m(k_j a)$ to get

$$-k_j a\frac{Y_m'(k_j a)}{Y_m(k_j a)}D_j + k_j b\frac{Y_m'(k_j b)}{Y_m(k_j a)}F_j + a\frac{Y_m(k_j a)}{Y_m(k_j a)}G_j - b\frac{Y_m(k_j b)}{Y_m(k_j a)}H_j = 0.$$

Then taking the limit as $k \to k_0 = 0$, and setting $j = 0$, we have

$$mD_0 - m\left(\frac{b}{a}\right)^{-m} F_0 + aG_0 - b\left(\frac{b}{a}\right)^{-m} H_0 = 0. \tag{9}$$

Similarly, dividing equation 7 by $J_m(k_j a)$ and taking the limit as $k \to k_0 = 0$, we get

$$-mD_0 + m\left(\frac{b}{a}\right)^{m} F_0 + aG_0 - b\left(\frac{b}{a}\right)^{m} H_0 = 0. \tag{10}$$

This gives us the system of linear equations

$$\begin{pmatrix} a & -b\left(\frac{b}{a}\right)^{-m} \\ -a & b\left(\frac{b}{a}\right)^{m} \end{pmatrix}\begin{pmatrix} G_0 \\ H_0 \end{pmatrix} = \begin{pmatrix} -mD_0 + m\left(\frac{b}{a}\right)^{-m} F_0 \\ -mD_0 + m\left(\frac{b}{a}\right)^{m} F_0 \end{pmatrix}.$$

This system has full rank for all positive integers m. This completes the sequences $(G_j)_{j\in\mathbb{Z}}$ and $(H_j)_{j\in\mathbb{Z}}$, solving the D to N map. By similar arguments, we can find a full-rank system of linear equations for both $j \neq 0$ and $j = 0$, followed by the desired sequences, in the case where the Neumann BCs are known and the Dirichlet BCs are unknown. The same applies for any other combination of two BCs known and two BCs unknown, or three BCs known and one BC unknown.

Now consider instead an IBVP with two known Robin BCs, denoted

$$q(a,t) + \gamma q_\rho(a,t) =: v(t),$$
$$q(b,t) + \delta q_\rho(b,t) =: w(t),$$

where none of f, g, d, h as previously defined are known, and $\gamma, \delta \in \mathbb{R}$. In addition to equations 7 and 8, we also obtain, by applying $\mathcal{F}_{\text{ser}}$ to t in the BCs,

$$D_j + \gamma G_j = V_j,$$
$$F_j + \delta H_j = W_j.$$

This gives us a system of four linear equations in four unknowns:

$$\begin{pmatrix} -k_j aJ_m'(k_j a) & k_j bJ_m'(k_j b) & aJ_m(k_j a) & -bJ_m(k_j b) \\ -k_j aY_m'(k_j a) & k_j bY_m'(k_j b) & aY_m(k_j a) & -bY_m(k_j b) \\ 1 & 0 & \gamma & 0 \\ 0 & 1 & 0 & \delta \end{pmatrix}\begin{pmatrix} D_j \\ F_j \\ G_j \\ H_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ V_j \\ W_j \end{pmatrix}.$$

The first two rows, R_1 and R_2, are linearly independent by definition of J_m and Y_m. Note that for $j \neq 0$, since the zeros of J_m' and Y_m' for $m \geqslant 0$ are all real, none of the entries in R_1 and R_2 are zero. Therefore, each of R_1 and R_2 is linearly independent from each of R_3 and R_4. Clearly, R_3 and R_4 are linearly independent. So this system has full rank. For $j = 0$, we make use of equations 9 and 10, and the Robin BC equations, to obtain the system

$$\begin{pmatrix} m & -m\left(\frac{b}{a}\right)^{m} & a & -b\left(\frac{b}{a}\right)^{m} \\ -m & m\left(\frac{b}{a}\right)^{-m} & a & -b\left(\frac{b}{a}\right)^{-m} \\ 1 & 0 & \gamma & 0 \\ 0 & 1 & 0 & \delta \end{pmatrix}\begin{pmatrix} D_0 \\ F_0 \\ G_0 \\ H_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ V_0 \\ W_0 \end{pmatrix}.$$

For positive integer m, all the rows are linearly independent and the system has full rank. So we can reconstruct d, f, g, h in terms of v, w for all $j \in \mathbb{Z}$.

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