Heat Equation with Dynamic Boundary Condition Reduces to Fractional Linear Ordinary Differential Equation

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Abstract

The heat equation is a partial differential equation that describes how heat is distributed in a region over time. In this project, we seek to solve the heat equation on the half line where the boundary condition at one end evolves with time using the Fokas method. We show that the problem reduces to a fractional linear ordinary differential equation (FLODE) with a variable coefficient. Drawing from ideas in fractional calculus, we then obtain a solution to the FLODE through the Frobenius method, thus solving the heat equation.

Introduction

Consider the following heat equation with dynamic boundary condition

$$\begin{aligned} q_t + q_{xx} &= 0, \qquad (x,t) \in (0,\infty) \times (0,T), \\ q(x,0) &= q_0(x), \qquad x \in [0,\infty), \\ q_x(0,t) + f(t)q(0,t) &= 0, \qquad t \in [0,T], \end{aligned}$$

where T is a positive constant.

Through the Fokas method, we find the solution to be given by

$$2\pi q(x,\tau) = \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 \tau} \hat{q}_0(\lambda)\, d\lambda - \int_{\partial D^+} e^{i\lambda x - \lambda^2 \tau} F(\lambda; T)\, d\lambda \tag{1}$$

where

$$F(\lambda; T) = \int_0^T e^{\lambda^2 s} q_x(0,s)\, ds + i\lambda \int_0^T e^{\lambda^2 s} q(0,s)\, ds,$$

and $D^+ = \{\lambda \in \mathbb{C}^+ : \Re(\lambda^2) < 0\}$.

Our goal is to express this solution in terms of known data. Through a process known as Dirichlet-to-Neumann Map, we are able to express the solution solely in terms of one boundary value, and reduce the problem to simply solving for that boundary value. We have effectively reduced the problem to solving just for $q(0,s)$ in

$$\hat{q}_0(-i\sqrt{-i\rho}) - = \int_0^T e^{i\rho s} (\sqrt{-i\rho} - f(s)) q(0,s)\, ds. \tag{2}$$

Fractional Integral and Fractional Derivative

Definition 1. *For $0 < \alpha < 1$, the Liouville left-sided fractional integral on $\mathbb{R}$ is defined as*

$$(I_+^\alpha y)(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{y(t)\, dt}{(x-t)^{1-\alpha}}, \tag{3}$$

Definition 2. *For $0 < \alpha < 1$, the Caputo derivative is defined as*

$$({}^C D_+^\alpha y)(x) := (I_+^{1-\alpha} D y)(x) \tag{4}$$

where $D = \frac{d}{dx}$.

Property 1. *For $0 < \alpha < 1$ and $\Re(\beta) > 1$,*

$$\left({}^C D_+^\alpha (t)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} (x)^{\beta-\alpha-1}. \tag{5}$$

In particular,

$$\left({}^C D_+^\alpha 1\right)(x) = 0$$

Fourier Transform of Fractional Integrals and Derivatves

Theorem 2. *Suppose q is a function in the Schwartz space such that*

$$q(s) = \begin{cases} q(s) & \text{if } s \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(\mathcal{F} I_+^\alpha q)(x) = \frac{\hat{q}(x)}{(-ix)^\alpha},$$

where $\hat{q}(x) = (\mathcal{F} q)(x)$.

Corollary 2.1. *Suppose that q and α are as the same in Theorem 2 and $q(0) = 0$, then*

$$(\mathcal{F}^C D_+^\alpha q)(x) = (-ix)^\alpha \hat{q}(x).$$

α-analyticity and Power Rule

Definition 3. *Let $\alpha \in (0,1]$ and $f(x)$ be a real function defined on some interval $[a,b]$ and $x_0 \in [a,b]$. Then f(x) is said to be α-analytic at x_0 if there exists an interval $N(x_0)$ such that for all $x \in N(x_0)$, f(x) can be expressed as $\sum_{n=0}^{\infty} a_n (x - x_0)^{n\alpha}$.*

Proposition 3. *Let $\alpha \in (0,1]$. If $f(x)$ is α-analytic at x_0, with convergence radius ρ, then*

$$({}^C D_+^\alpha f)(x) = \left({}^C D_{a+}^\alpha \left(\sum_{n=0}^{\infty} a_n (t - x_0)^{n\alpha}\right)\right)(x) = \sum_{n=0}^{\infty} a_n ({}^C D_{a+}^\alpha (t - x_0)^{n\alpha})(x).$$

Theorem 4. *Let $\alpha \in (0,1]$, and let $f(x) = q(x)$ where $q(x)$ is as defined in Theorem 2 and such that $f(x) = \sum_{n=0}^{\infty} a_n x^{n\alpha}$. Then,*

$$({}^C D_+^\alpha f)(x) = \sum_{n=1}^{\infty} a_n \frac{\Gamma(n\alpha + 1)}{\Gamma((n-1)\alpha + 1)} x^{(n-1)\alpha}.$$

Solving the Fractional Linear Ordinary Differential Equation via the method of Frobenius

By taking the Fourier inverse of the RHS of Equation 2 and using Theorem 2 ,Equation 2 reduces to a Fractional Linear Ordinary Differential Equation of the form

$$({}^C D_+^{1/2} y)(t) - f(t) y(t) = g(t) \tag{6}$$

where $y(t) = q(0,t)$ and $g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\rho t} \hat{q}_0(-i\sqrt{-i\rho})\, d\rho$. Suppose that $y(t)$ is α-analytic about the α-ordinary point 0. We seek the series solution of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^{n/2}.$$

By Theorem 4,

$$({}^C D_+^{1/2} y)(t) = \sum_{n=1}^{\infty} a_n \frac{\Gamma(n/2 + 1)}{\Gamma((n-1)/2 + 1)} t^{(n-1)/2}.$$

Further suppose that $f(t)$ and $g(t)$ are also α-analytic about 0, i.e $f(t) = \sum_{n=0}^{\infty} b_n t^{n/2}$ and $g(t) = \sum_{n=0}^{\infty} c_n t^{n/2}$. We can express the coefficients a_{n+1} in terms of a_0, b_n and c_n by the following recurrence relation

$$a_{n+1} = \frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{n+3}{2})} \left(\sum_{k=0}^{n} a_k b_{n-k} + c_n\right)$$

with $a_0 = 0$ by necessity of Corollary 2.1. We can thus compute the coefficients of $q(0,t)$, which will then lead us to the solution of the heat equation.

Plots of Solutions to FLODE

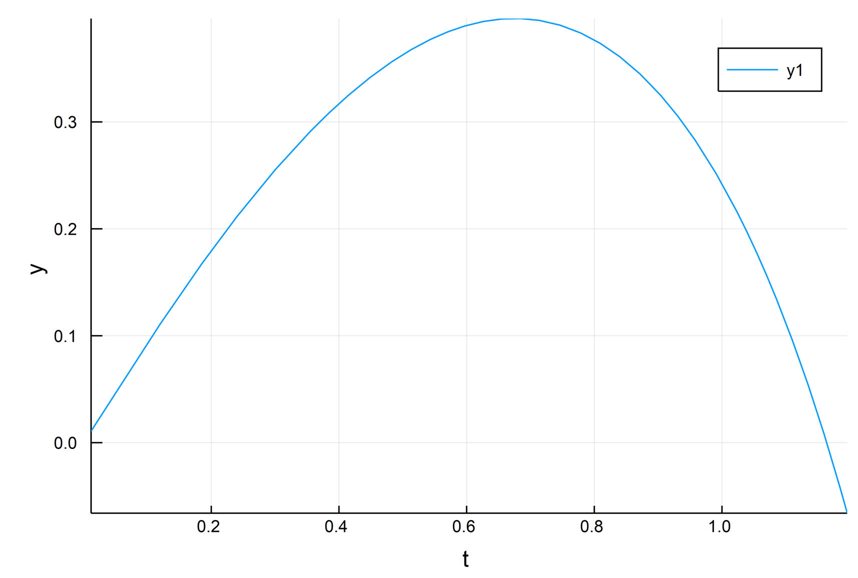

Figure 1:Plot of $y(t)$ at 50^{th} order of approximation where $f(t) = t^{\frac{1}{2}} - \frac{1}{2}t + \frac{1}{4}t^{\frac{3}{2}} + \frac{1}{8}t^2$ and $g(t) = t^{\frac{1}{2}} + \frac{1}{2}t - 2t^{\frac{3}{2}} + \frac{1}{8}t^2$

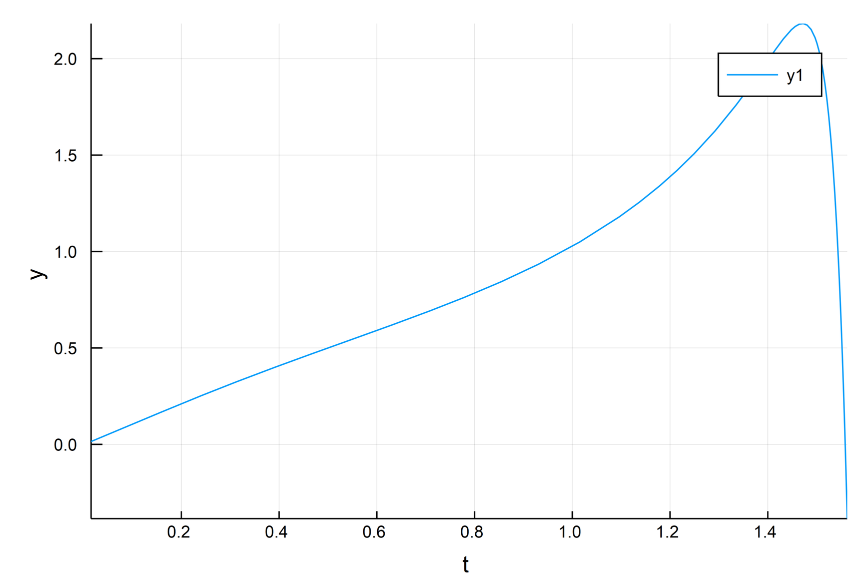

Figure 2:Plot of $y(t)$ at 50^{th} order of approximation where $f(t) = t^{\frac{1}{2}} - 2t + \frac{1}{4}t^{\frac{3}{2}} + t^2$ and $g(t) = t^{\frac{1}{2}} + t - 2t^{\frac{3}{2}} + t^2$

Applications of Fractional Differential Equations

In systems where anomalous dynamics are present, fractional differential equations are more accurate than differential equations with classical derivatives in modelling anomalous processes. For example, the Porous Medium Equation (PME) which models non-linear heat flow, and gas flow in porous medium, has been extended into fractional forms to account for anomalous diffusion which then have concrete applications such as in the study of moisture dispersion in porous construction materials.

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