Fokas Method for Heat Equation with Dynamic Boundary Condition

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Abstract

This report aims to derive the integral representation of the solution to the heat equation with dynamic Robin condition via the Fokas method. We begin with a construction of the solution representation via the Fokas method and show that the problem reduces to a fractional linear ordinary differential equation (FLODE). We then give a brief overview of Caputo fractional derivative and solve the FLODE via the Frobenius Method. Finally, we provide an error estimation via the generalised Taylor's Remainder Theorem.

1 Introduction

The Fokas method is a method for solving linear evolution partial differential equations with spatial derivatives of arbitray order of the form

$$(\partial_t + w(-i\partial_x))q(x,t) = 0, \quad 0 < x < \infty, \quad t > 0, \tag{1.1}$$

where $w(\lambda)$ is an arbitary polynomial. In particular, the heat equation given by the partial differential equation

$$q_t - q_{xx} = 0 \tag{1.2}$$

is specified by $w(\lambda) = \lambda^2$.

The Fokas method proceeds by first rewriting Equation (1.1) in the form

$$X_t - Y_x = 0, \quad 0 < x < \infty, \quad t > 0, \tag{1.3}$$

where X and Y is defined as

$$\begin{aligned} X(x,t,\lambda) &= e^{-i\lambda x + w(\lambda)t} q(x,t), \\ Y(x,t,\lambda) &= e^{-i\lambda x + w(\lambda)t} H(x,t,\lambda) \end{aligned}$$

and $H(x,t,\lambda)$ is given by the formula

$$H = i\frac{w(-i\partial)_x - w(\lambda)}{-i\partial_x - \lambda}. \tag{1.4}$$

From Equation (1.3), we can obtain the Global Relation and the integral representation of the solution. This will be elaborated in the following section.

1.1 Heat Equation with Dynamic Boundary Condition

Consider the following heat equation with dyanamic boundary condition

$$\begin{aligned} &q_t + q_{xx} = 0, \qquad (x,t) \in (0,\infty) \times (0,T), \\ &q(x,0) = q_0(x), \qquad x \in [0,\infty), \\ &q_x(0,t) + f(t)q(0,t) = 0, \qquad t \in [0,T] \end{aligned}$$

where T is a positive constant. From Equation (1.4), we define

$$X(x,t,\lambda) = \left(e^{-i\lambda x+\lambda^2 t} q(x,t)\right) \tag{1.5}$$

$$Y(x,t,\lambda) = \left(e^{-i\lambda x+\lambda^2 t} (\partial_x + i\lambda) q(x,t)\right). \tag{1.6}$$

The heat equation can then be rewritten as

$$X_t - Y_x = 0. \tag{1.7}$$

We can check this by substituting X and Y into (1.7) and obtaining

$$e^{-i\lambda x+\lambda^2 t}(q_t - q_{xx}) = 0,$$

which is true iff $q_t - q_{xx} = 0$.

1.1.1 Global Relation and Fourier Representation of Solution

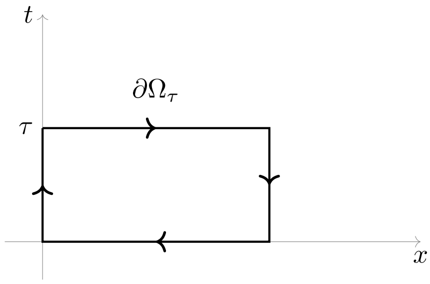

Figure 1: Green's Theorem on Ω_τ

We apply Green's theorem to Equation (1.7) on the region $\Omega_\tau : (0,\infty) \times (0,\tau), \tau \in [t,T]$ as show in Figure 1 to obtain

$$\begin{aligned}
\int_{\Omega_\tau} (X_t - Y_x)\,dx\,dt =& \int_{\partial\Omega_\tau} (Y\,dt + X\,dx) \\
=& \int_0^\tau Y(0,s,\lambda)\,ds - \int_0^\tau \lim_{x\to\infty} Y(x,s,\lambda) \\
&- \int_0^\infty X(x,0,\lambda)\,dx + \int_0^\infty X(x,\tau,\lambda)\,dx \\
=& \int_0^\tau e^{\lambda^2 s} q_x(0,s)\,ds + i\lambda \int_0^\tau e^{\lambda^2 s} q(0,s) ds \\
&- \int_0^\infty e^{-i\lambda x} q(x,0)\,dx + \int_0^\infty e^{-i\lambda x+\lambda^2\tau} q(x,\tau)\,dx \\
=& 0.
\end{aligned}$$

By defining

$$F(\lambda;\tau) = \int_0^\tau e^{\lambda^2 s} q_x(0,s)\,ds + i\lambda \int_0^\tau e^{\lambda^2 s} q(0,s)\,ds,$$

we obtain the Global Relation

$$F(\lambda;\tau) - \hat{q}_0(\lambda) + e^{\lambda^2\tau}\hat{q}(\lambda;\tau) = 0 \tag{1.8}$$

which relates the solution to the boundary values and initial conditions. By performing Fourier Inverse Transform on Equation (1.8), we obtain the Fourier representation of the solution

$$2\pi q(x,\tau) = \int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2\tau}\hat{q}_0(\lambda)\,d\lambda - \int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2\tau} F(\lambda;\tau)\,d\lambda. \tag{1.9}$$

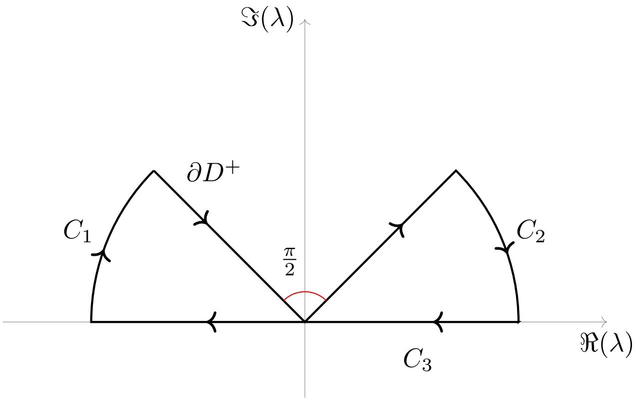

Figure 2: Deform integration contour into complex plane

1.1.2 Contour Deformation

We define the region $D^+ = \{\lambda \in \mathbb{C}^+ : \Re(\lambda^2) < 0\}$. Since $e^{i\lambda x - \lambda^2\tau} F(\lambda;\tau)$ is analytic in λ, by Cauchy's Theorem,

$$\oint\limits_{C_1+\partial D^+ + C_2 + C_3} e^{i\lambda x - \lambda^2\tau} F(\lambda;\tau)\, d\lambda = 0. \tag{1.10}$$

We define I by

$$I := \int\limits_{\partial(\mathbb{C}^+ \setminus D^+)} e^{i\lambda x - \lambda^2\tau} F(\lambda;0)\, d\lambda.$$

For $\tau = 0$, $F(\lambda;0) = 0$ and hence $I = 0$. Assume $\tau > 0$, then

$$\begin{aligned} e^{-\lambda^2\tau} F(\lambda;\tau) &= e^{-\lambda^2\tau} \int_0^\tau e^{\lambda^2 s} (i\lambda + \partial_x) q(0,s)\, ds \\ &= e^{-\lambda^2 t} \left\{ \left[\frac{e^{\lambda^2 s}}{\lambda^2} (i\lambda + \partial_x) q(0,s) \right]_{s=0}^{s=t} - \frac{1}{\lambda^2} \int_0^t e^{\lambda^2 s} \partial_s (i\lambda + \partial_x) q(0,s)\, ds \right\} \\ &= \mathcal{O}(\lambda^{-1}) \end{aligned}$$

due to the $i\lambda$ factor. By Jordan's Lemma,

$$\left(\int\limits_{C_1} + \int\limits_{C_2} \right) e^{i\lambda x - \lambda^2\tau} F(\lambda;\tau)\, d\lambda = 0.$$

Then, from Equation 1.10, we conclude that

$$\int\limits_{\partial D^+} e^{i\lambda x - \lambda^2\tau} F(\lambda;\tau)\, d\lambda = - \int\limits_{C_3} e^{i\lambda x - \lambda^2\tau} F(\lambda;\tau)\, d\lambda.$$

Hence, we can deform the integration contour into the complex plane and Equation (1.9) can be written as

$$2\pi q(x,\tau) = \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2\tau} \hat{q}_0(\lambda)\, d\lambda - \int\limits_{\partial D^+} e^{i\lambda x - \lambda^2\tau} F(\lambda;\tau)\, d\lambda \tag{1.11}$$

1.1.3 Replacing $F(\lambda;\tau)$ with $F(\lambda;T)$

Since $e^{i\lambda x - \lambda^2\tau}(F(\lambda;\tau) - F(\lambda;T))$ is analytic in λ, by Cauchy's Theorem,

$$\oint\limits_{\partial D^+ + C_1} e^{i\lambda x - \lambda^2\tau} (F(\lambda;\tau) - F(\lambda;T))\, d\lambda = 0. \tag{1.12}$$

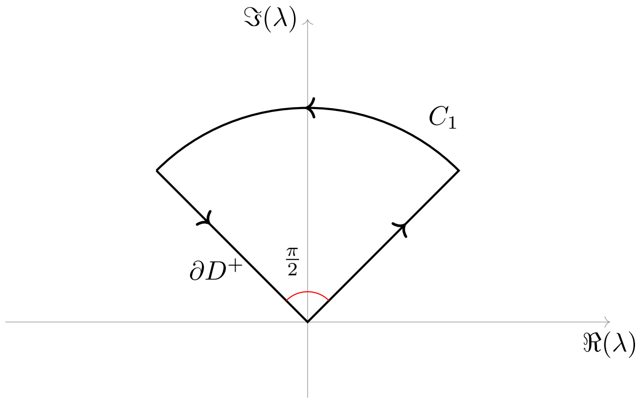

Figure 3: Jordan's Lemma for $F(\lambda;\tau)$

Consider on C_1 the equation

$$\int_{\partial D^+} e^{i\lambda x-\lambda^2\tau}(F(\lambda;\tau)-F(\lambda;T))\,d\lambda. \tag{1.13}$$

Following the same arugment as above, we integrate by parts Equation 1.13

$$\begin{aligned}
e^{-\lambda^2\tau}(F(\lambda;\tau)-F(\lambda;T)) &= e^{-\lambda^2\tau}\left[\int_0^\tau e^{\lambda^2 s}(i\lambda+\partial_x)q(0,s)\,ds - \int_0^T e^{\lambda^2 s}(i\lambda+\partial_x)q(0,s)\,ds\right] \\
&= e^{-\lambda^2 t}\left\{\left[\frac{e^{\lambda^2 s}}{\lambda^2}(i\lambda+\partial_x)q(0,s)\right]_{s=0}^{s=\tau} - \frac{1}{\lambda^2}\int_0^\tau e^{\lambda^2 s}\partial_s(i\lambda+\partial_x)q(0,s)\,ds\right\} \\
&\quad - e^{-\lambda^2 t}\left\{\left[\frac{e^{\lambda^2 s}}{\lambda^2}(i\lambda+\partial_x)q(0,s)\right]_{s=0}^{s=T} - \frac{1}{\lambda^2}\int_0^T e^{\lambda^2 s}\partial_s(i\lambda+\partial_x)q(0,s)\,ds\right\} \\
&= \mathcal{O}(\lambda^{-1}).
\end{aligned}$$

Hence, by Jordan's Lemma, from Equation 1.12 we conclude that

$$\int_{C_1} e^{i\lambda x-\lambda^2\tau}(F(\lambda;\tau)-F(\lambda;T))\,d\lambda = -\int_{\partial D^+} e^{i\lambda x-\lambda^2\tau}(F(\lambda;\tau)-F(\lambda;T))\,d\lambda = 0.$$

This implies that we can use $F(\lambda;T)$ instead in our solution, i.e

$$2\pi q(x,\tau) = \int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2\tau}\hat{q}_0(\lambda)\,d\lambda - \int_{\partial D^+} e^{i\lambda x-\lambda^2\tau}F(\lambda;T)\,d\lambda \tag{1.14}$$

which further implies that our solution does not depend on the part from τ to T.

1.1.4 Dirichlet-to-Neumaan Map

Recall that in our Fourier representation of the solution, it contains 2 two boundary values. Our aim is to express the solution in terms of only one boundary value and solve for it. We first apply the time transform to the dynamic boundary condition to obtain

$$\int_0^\tau e^{\lambda^2 s}q_x(0,s)\,ds + \int_0^\tau e^{\lambda^2 s}f(s)q(0,s)\,ds = 0. \tag{1.15}$$

Since the time transform $F(\lambda;0)$ relies on λ^2, we can get another equation from the Global Relation

$$F(-\lambda;0) - \hat{q}_0(-\lambda) + e^{\lambda^2\tau}\hat{q}(-\lambda;\tau) = 0 \tag{1.16}$$

$$\Rightarrow \int_0^\tau e^{\lambda^2 s}q_x(0,s)\,ds + i\lambda\int_0^\tau e^{\lambda^2 s}q(0,s)\,ds = \hat{q}_0(-\lambda) - e^{\lambda^2\tau}\hat{q}(-\lambda;\tau). \tag{1.17}$$

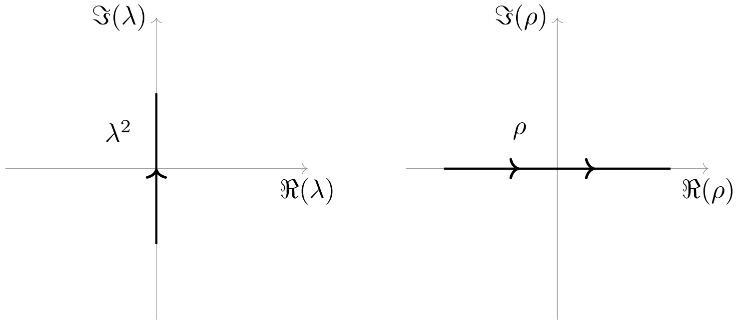

Figure 4: Change of variable

Subtracting Equation (1.17) by Equation (1.15) , we obtain

$$\hat{q}_0(-\lambda) - e^{\lambda^2\tau}\hat{q}(-\lambda;\tau) = \int_0^\tau e^{\lambda^2 s}(-i\lambda - f(s))q(0,s)\,ds. \tag{1.18}$$

By a change of variables $\lambda = i\sqrt{-i\rho}$ where ρ is the real line in the positive direction and letting $\tau = T$, we obtain the equation

$$\hat{q}_0(-i\sqrt{-i\rho}) - e^{i\rho T}\hat{q}(-i\sqrt{-i\rho};T) = \int_0^T e^{i\rho s}(\sqrt{-i\rho} - f(s))q(0,s)\,ds. \tag{1.19}$$

By taking the Fourier inverse of the RHS of Equation 1.19, we then obtain the fractional linear ordinary differential equation (FLODE)

$$({}^C D_+^{1/2} y)(t) - f(t)y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\rho t}\left[\hat{q}_0(-i\sqrt{-i\rho}) - e^{i\rho T}\hat{q}(-i\sqrt{-i\rho};T)\right]\,d\rho \tag{1.20}$$

where $y(t) = q(0,t)$. The justification for the FLODE will be shown in Section 2.

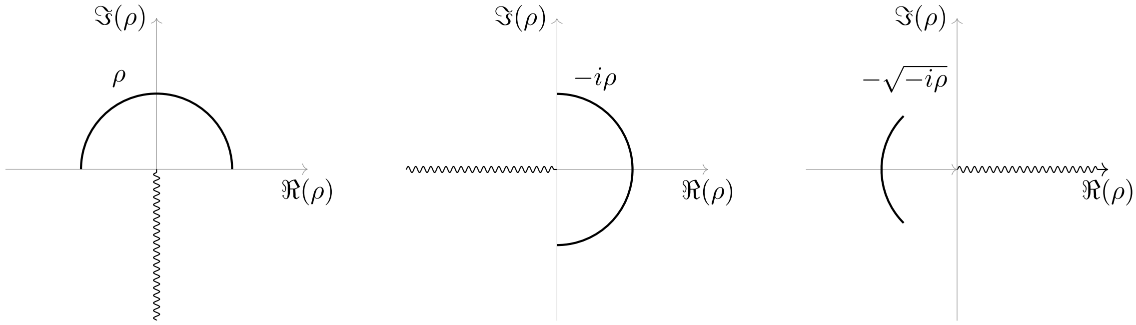

Figure 5: $\Re(-\sqrt{-i\rho}) < 0$

We will now show that $\displaystyle\int_{-\infty}^{\infty} -e^{i\rho T}\hat{q}(-i\sqrt{-i\rho};T)\,d\rho = 0$. Consider $\hat{q}(-i\sqrt{-i\rho};T)$ on the upper half of the complex ρ plane, where a branch cut is taken on the negative imaginary axis with a continuous branch point at 0. Since $\Re(-i\sqrt{-i\rho}) < 0$ on this plane, $\hat{q}(-i\sqrt{-i\rho};T)$ is bounded. Moreover,

$$\begin{aligned}
|\hat{q}(-i\sqrt{-i\rho};T)| &= \left|\int_0^\infty e^{-\sqrt{-i\rho}x}q(x;T)\,dx\right| \\
&= \left|\left[-\frac{1}{\sqrt{-i\rho}}e^{-\sqrt{-i\rho}x}q(x;T)\right]_{x=0}^{x=\infty} + \int_0^\infty \frac{1}{\sqrt{-i\rho}}e^{-\sqrt{-i\rho}x}q_x(x;T)\,dx\right| \\
&= \mathcal{O}\left(\frac{1}{|\sqrt{-i\rho}|}\right)
\end{aligned}$$

which decays to 0 as $|\rho| \to \infty$. Hence, by Jordan's Lemma, $\lim_{R\to\infty} \int_{C_R} -e^{i\rho T}\hat{q}(-i\sqrt{-i\rho};T)\,d\rho = 0$. Furthermore, note that $-e^{i\rho T}\hat{q}(-i\sqrt{-i\rho};T)$ is analytic on the closed upper half plane except at 0, but continuous at 0, and hence, by the Extended Cauchy-Goursat Theorem, $\oint_C -e^{i\rho T}\hat{q}(-i\sqrt{-i\rho};T)\,d\rho = 0$ which implies that $\lim_{R\to\infty} \int_{-R}^{R} -e^{i\rho T}\hat{q}(-i\sqrt{-i\rho};T)\,d\rho = 0$. We can thus rewrite Equation 1.20 as

$$({}^C D_+^{1/2} y)(t) - f(t)y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\rho t}\hat{q}_0(-i\sqrt{-i\rho})\,d\rho. \tag{1.21}$$

2 Fractional Differential Equation

2.1 Liouville Fractional Integral and Caputo Fractional Derivative

For $0 < \alpha < 1$, the Liouville left-sided fractional integrals on $\mathbb{R}$ is defined by

$$(I_+^\alpha y)(x) := \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x} \frac{y(t)\,dt}{(x-t)^{1-\alpha}}, \tag{2.1}$$

while the Caputo fractional derivative is defined by

$$({}^C D_+^\alpha y)(x) := (I_+^{1-\alpha} D y)(x) \tag{2.2}$$

where $D = \frac{d}{dx}$ denotes the usual first derivative.

2.2 Fourier Transform of Caputo Fractional Derivatives

Theorem 2.1 (Samko [4]). *Suppose q is a function in the Schwartz space such that*

$$q(s) = \begin{cases} q(s) & \text{if } s \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$(\mathcal{F} I_+^\alpha q)(x) = \frac{\hat{q}(x)}{(-ix)^\alpha}$$

where $\hat{q}(x) = (\mathcal{F}q)(x)$.

Proof. Note that

$$(\mathcal{F}q)(x) = \frac{d}{dx}\int_{-\infty}^{\infty} \frac{e^{ixt}-1}{it} q(t)\,dt. \tag{2.3}$$

Since $q(s) = 0$ for $s < 0$, by Equations (2.1) and (2.3),

$$(\mathcal{F} I_+^\alpha q)(x) = \frac{1}{\Gamma(\alpha)}\frac{d}{dx}\int_0^\infty \frac{e^{ixt}-1}{it}\int_0^t \frac{q(s)}{(t-s)^{1-\alpha}}\,ds\,dt.$$

By Fubini's Theorem, we can interchange the order of integration and thus,

$$\begin{aligned}
(\mathcal{F}I_+^\alpha q)(x) &= \frac{1}{\Gamma(\alpha)}\frac{d}{dx}\int_0^\infty \frac{e^{ixt}-1}{it}\int_0^t \frac{q(s)}{(t-s)^{1-\alpha}}\,ds\,dt \\
&= \frac{1}{\Gamma(\alpha)}\int_0^\infty q(s)\int_s^\infty \frac{d}{dx}\frac{e^{ixt}-1}{it(t-s)^{1-\alpha}}\,dt\,ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^\infty q(s)\int_s^\infty \frac{e^{ixt}}{(t-s)^{\alpha}}\,dt\,ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^\infty q(s)\int_s^\infty \frac{e^{ix(\tau+s)}}{\tau^{1-\alpha}}\,d\tau\,ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^\infty q(s)e^{ixs}\,ds\int_s^\infty \frac{e^{ix(\tau)}}{\tau^{1-\alpha}}\,d\tau \\
&= \frac{\hat{q}(x)}{\Gamma(\alpha)}\Gamma(\alpha)(-ix)^{-\alpha} \\
&= \frac{\hat{q}(x)}{(-ix)^\alpha}
\end{aligned}$$

where the third last equality is given by a change of variable $\tau = t - s$. □

Corollary 2.1.1. *Provided that q and α is the same as in Theorem 2.1 and $q(0) = 0$, then*

$$(\mathcal{F}{}^C D_+^\alpha q)(x) = (-ix)^\alpha \hat{q}(x).$$

Proof. Note that

$$\begin{aligned}
\hat{q'}(x) &= \lim_{s\to\infty}\int_{-s}^s q'(t)e^{ixt}\,dt \\
&= \lim_{s\to\infty}\int_0^s q'(t)e^{ixt}\,dt \\
&= \lim_{s\to\infty}\left[q(t)e^{ixt}\right]_0^s - ix\int_0^\infty q(t)e^{ixt}dt \\
&= -q(0) - ix\hat{p}(x).
\end{aligned}$$

Since $q(0) = 0$, by Equation 2.2 and Theorem 2.1,

$$\begin{aligned}
(\mathcal{F}{}^c D_+^\alpha q)(x) &= (\mathcal{F}I_+^{1-\alpha} q')(x) \\
&= \frac{\hat{q'}(x)}{(-ix)^{1-\alpha}} \\
&= (-ix)^\alpha \hat{q}(x).
\end{aligned}$$

□

2.3 α-analyticity and Power Rule

Definition 2.1. *Let $\alpha \in (0,1]$ and $f(x)$ be a real function defined on some interval $[a,b]$ and $x_0 \in [a,b]$. Then f(x) is said to be α-analytic at x_0 if there exists an interval $N(x_0)$ such that for all $x \in N(x_0)$, f(x) can be expressed as $\sum_{n=0}^\infty a_n(x-x_0)^{n\alpha}$.*

Theorem 2.2. *Let $\alpha \in (0,1]$. If $f(x)$ is α-analytic at x_0, with convergence radius R, then*

$$({}^C D_+^\alpha f)(x) = \left({}^C D_{a+}^\alpha\Big(\sum_{n=0}^\infty a_n(t-x_0)^{n\alpha}\Big)\right)(x) = \sum_{n=0}^\infty a_n({}^C D_{a+}^\alpha (t-x_0)^{n\alpha})(x).$$

Theorem 2.3. *Let $\alpha \in (0,1]$, and let $f(x) = q(x)$ where $q(x)$ is defined in Theorem 2.1. Further suppose that $f(x)$ is α-analytic, i.e $f(x) = \sum_{n=0}^\infty a_n x^{n\alpha}$. Then,*

$$({}^C D_+^\alpha f)(x) = \sum_{n=1}^\infty a_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} x^{(n-1)\alpha}.$$

Proof. Note that $f(x) = 0$ for all $x < 0$. Then, by Equation 2.2,

$$\begin{aligned}
({}^C D_+^\alpha f)(x) &= \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{f'(t)}{(x-t)^\alpha}\, dt \\
&= \frac{1}{\Gamma(1-\alpha)} \int_{0}^{x} \frac{f'(t)}{(x-t)^\alpha}\, dt \\
&= \frac{1}{\Gamma(1-\alpha)} \int_{0}^{x} (x-t)^{-\alpha} \left(\sum_{n=1}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha)} t^{n\alpha-1} \right) dt \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{n=1}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha)} \int_{0}^{x} (x-t)^{-\alpha} (t^{n\alpha-1})\, dt \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{n=1}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha)} \int_{0}^{1} (x-xv)^{-\alpha} (xv)^{n\alpha-1} x\, dv \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{n=1}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha)} x^{(n-1)\alpha} \int_{0}^{1} (1-v)^{-\alpha} (v)^{n\alpha-1}\, dv \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{n=1}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha)} x^{(n-1)\alpha} B(n\alpha, 1-\alpha) \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{n=1}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha)} x^{(n-1)\alpha} \frac{\Gamma(n\alpha)\Gamma(1-\alpha)}{\Gamma((n-1)\alpha+1)} \\
&= \sum_{n=1}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} x^{(n-1)\alpha}.
\end{aligned}$$

□

N.B. The original proof by El-Ajou [1] was for Caputo derivatives on a finite interval, but since our function $q(x)$ is 0 on the left half-line, we are able to get the same property.

Theorem 2.4. *Suppose that f has a fractional power series representation of the form*

$$f(t) = \sum_{n=0}^{\infty} a_n (t-t_0)^{n\alpha}, 0 \le m-1 < \alpha \le m, t_0 \le t < t_0 + R.$$

If $f(t) \in C[t_0, t_0+R)$ and ${}^C D_{t_0}^{n\alpha} \in C(t_0, t_0+R)$ for $n = 0, 1, 2, \ldots,$ then the coefficients a_n will take the form $a_n = \dfrac{{}^C D_{t_0}^{n\alpha} f(t_0)}{\Gamma(n_\alpha + 1)}$, where ${}^C D_{t_0}^{n\alpha} = {}^C D_{t_0}^{\alpha} \cdot {}^C D_{t_0}^{\alpha} \cdot \ldots \cdot {}^C D_{t_0}^{\alpha} (n-times)$.

Proof. Assume that f is an arbitrary function that can be represented by a fractional power series expansion. Let $t = t_0$, then $f(t_0) = a_0$, and the remaining terms vanishes. From Theorem 2.3,

$$ {}^C D_{t_0}^{\alpha} f(t) = \sum_{n=1}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} (t-t_0)^{(n-1)\alpha} \tag{2.4}$$

and let $t = t_0$. Then, $c_1 \Gamma(\alpha+1) = {}^C D_{t_0}^{\alpha} f(t_0)$. Further apply Theorem 2.3 to Equation 2.4, we find that

$$ {}^C D_{t_0}^{2\alpha} f(t) = \sum_{n=2}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-2)\alpha+1)} (t-t_0)^{(n-2)\alpha}, \tag{2.5}$$

and ${}^C D_{t_0}^{2\alpha} f(t_0) = a_2 \Gamma(2\alpha+1)$. By repeated application of Theorem 2.3, and letting $t = t_0$, we find the general formula for $a_n = \dfrac{{}^C D_{t_0}^{n\alpha} f(t_0)}{\Gamma(n_\alpha + 1)}$. □

N.B Again, might not be true in general for caputo derivatives on left half-line, since this theorem is dependent on Theorem 2.3

Corollary 2.4.1. *Suppose f is a function with a fractional power series expansion as in Theorem 2.4, then the expansion is precisely the Generalised Taylor's Series.*

Proof. Substitute the general formula for a_n in the fractional series representation of f to obtain

$$f(t) = \sum_{n=0}^{\infty} \frac{{}^C D_{t_0}^{n\alpha} f(t_0)}{\Gamma(n\alpha + 1)} (t - t_0)^{n\alpha}.$$

□

Let $T_n(t) := \sum_{j=o}^{n} \frac{{}^C D_0^{j\alpha} f(t_0)}{\Gamma(j\alpha+1)} (t - t_0)^{j\alpha}$ be the n-th degree Taylor polynomial of f. If f is analytic, then $f(t) = \lim_{n\to\infty} T_n(t)$ as shown. We define the remainder by $R_n(t) = f(t) - T_n(t)$

Theorem 2.5 (Remainder Theorem of Generalised Taylor's Series [1])**.** *Suppose that f is $\alpha-$analytic. If $\left|{}^C D_{t_0}^{(n+1)\alpha} f(t)\right| \leq M$ on $t_0 \leq t \leq d$ where $0 < \alpha \leq 1$, then the remainder $R_n(t)$ satisfies the inequality*

$$|R_n(t)| \leq \frac{M}{\Gamma((n+1)\alpha + 1)} (t - t_0)^{(n+1)\alpha}, t_0 \leq t \leq d.$$

2.4 Frobenius Method to Solve FLODE

Recall that we are trying to solve for $y(t)$ in Equation 1.21

$$({}^C D_+^{1/2} y)(t) - f(t)y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\rho t} \hat{q}_0(-i\sqrt{-i\rho})\, d\rho := g(t). \tag{2.6}$$

Suppose that $y(t)$ is α-analytic about the α-ordinary point 0 with radius R. We seek the series solution of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^{n/2}.$$

By Theorem 2.3,

$$({}^C D_+^{1/2} y)(t) = \sum_{n=1}^{\infty} a_n \frac{\Gamma(n/2 + 1)}{\Gamma((n-1)/2 + 1)} t^{(n-1)/2}.$$

Further suppose that $f(t)$ and $g(t)$ are also α-analytic about 0 with radius R, i.e $f(t) = \sum_{n=0}^{\infty} b_n t^{n/2}$ and $g(t) = \sum_{n=0}^{\infty} c_n t^{n/2}$, where $t \in$ (0, We can thus express the coefficients a_{n+1} in terms of a_0, b_n and c_n by the following recurrence relation

$$a_{n+1} = \frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{n+3}{2})} \left(\sum_{k=0}^{n} a_k b_{n-k} + c_n \right)$$

with $a_0 = 0$ by necessity of Corollary 2.1.1.

3 Error Estimation

We found the integral representation of the solution to be given by

$$2\pi q(x,\tau) = \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 \tau} \hat{q}_0(\lambda)\, d\lambda - \int_{\partial D^+} e^{i\lambda x - \lambda^2 \tau} F(\lambda; T)\, d\lambda$$

where

$$\begin{aligned} F(\lambda;\tau) &= \int_0^{\tau} e^{\lambda^2 s} q_x(0,s)\, ds + i\lambda \int_0^{\tau} e^{\lambda^2 s} q(0,s)\, ds \\ &= \int_0^{\tau} (i\lambda - f(s)) e^{\lambda^2 s} q(0,s)\, ds. \end{aligned}$$

Since

$$q(0,s) := y(s) = \sum_{n=1}^{\infty} a_n s^{n/2} = \sum_{n=1}^{N} a_n s^{n/2} + R_N(s),$$

from Section 2.4, the error is given by

$$E(x,\tau) := \int\limits_{\partial D^+} e^{i\lambda x - \lambda^2 \tau} \int_0^\tau (i\lambda - f(s)) e^{\lambda^2 s} R_N(s)\, ds\, d\lambda. \tag{3.1}$$

We aim to estimate this error by restricting to $-1 \leq \Re(\partial D^+) \leq 1$. By Theorem 2.5, we find that if $\left| {}^C D_0^{(N+1)/2} y(s) \right| \leq M$, then $|R_N(s)| \leq \frac{M}{\Gamma(\frac{N+3}{2})} s^{(N+1)/2}$. So,

$$|E_{est}(x,\tau)| = \left| \int\limits_{|\Re(\partial D^+)| \leq 1} e^{i\lambda x - \lambda^2 \tau} \int_0^\tau (i\lambda - f(s)) e^{\lambda^2 s} R_N(s)\, ds\, d\lambda \right|$$

Let $g(x,\tau) = e^{i\lambda x - \lambda^2 \tau} \int_0^\tau (i\lambda - f(s)) e^{\lambda^2 s} R_N(s)\, ds$, $N = 2M - 1$, and $k = \frac{M}{\Gamma(\frac{N+3}{2})}$. Furthermore, assume $|f(s)| \leq L$ for some $L \in \mathbb{R}_{\geq 0}$. We seek a bound for g on the contour.

$$\begin{aligned}
|g(x,\tau)| &= \left| e^{i\lambda x - \lambda^2 \tau} \int_0^\tau (i\lambda - f(s)) e^{\lambda^2 s} R_N(s)\, ds \right| \\
&\leq \left| e^{i\lambda x - \lambda^2 \tau} \right| \int_0^\tau \left| (i\lambda - f(s)) e^{\lambda^2 s} R_N(s) \right| \, ds \\
&\leq e^{-\Im(\lambda) x} \int_0^\tau |(i\lambda - f(s))| \left| e^{\lambda^2 s} R_N(s) \right| \, ds \\
&\leq e^{-\Im(\lambda) x} \int_0^\tau (|\lambda| + |f(s)|) \left| R_N(s) \right| \, ds \text{ (since } \Re(\lambda^2) = 0) \\
&\leq e^{-\Im(\lambda) x} \int_0^\tau (|\lambda| + L) k s^m \, ds \\
&= e^{-\Im(\lambda) x} \frac{(|\lambda| + L)k}{m+1} \tau^{m+1} \\
&\leq \frac{(1+L)k}{m+1} \tau^{m+1}.
\end{aligned}$$

Since the contour is of length $2\sqrt{2}$, by the M-L estimation lemma,

$$|E_{est}(x,\tau)| \leq \frac{2(1+L)k}{\sqrt{2}(m+1)} \tau^{m+1}.$$

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